

Welcome to T35A

- Three Handouts
- Today:
 - Course Overview
 - Introduction to Set Theory
 - The Limits of Computation

The Course Website

Course Mailing :
teoriacomputacional@uaslp.edu.mx
<http://carlosgi.work/tc/>

Goals for this Course

- Explore **mathematical structures** that arise in math and computing.
- Equip you with the **fundamental mathematical tools** to reason about problems that arise in computing.
- Explore the **limits of computing** and what can be computed.
- Explore the **inherent complexity** of problems and why some problems are harder than others.

Introduction to Set Theory

“Cool people”

“The chemical elements”

“Cute animals”

“US coins.”

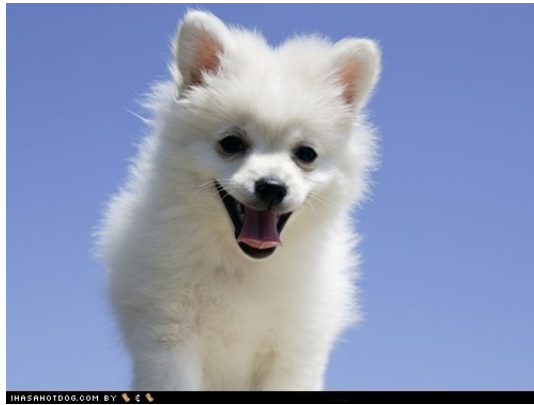
A **set** is an unordered collection of distinct objects, which may be anything (including other sets).



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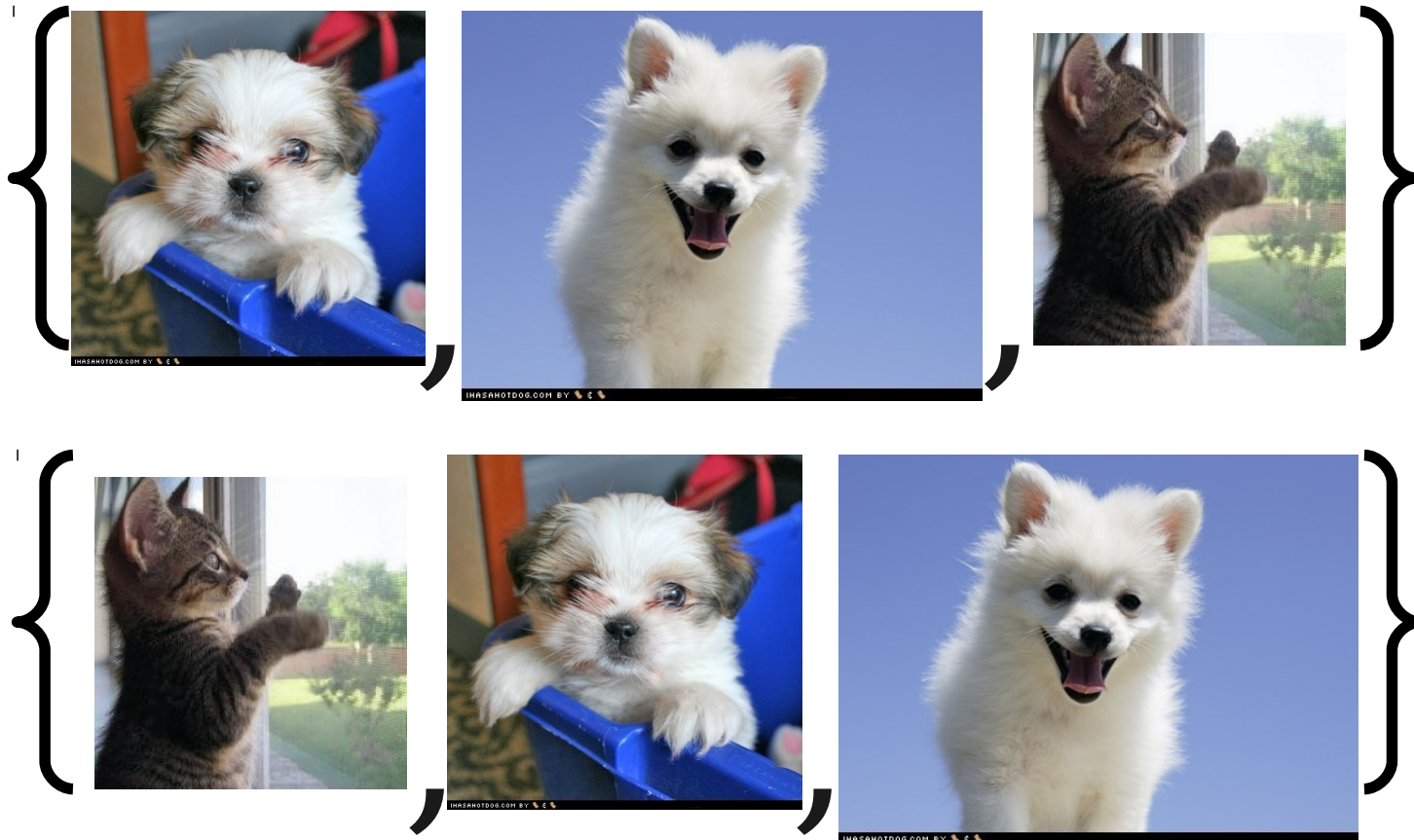


Set notation: Curly braces
with commas separating out
the elements

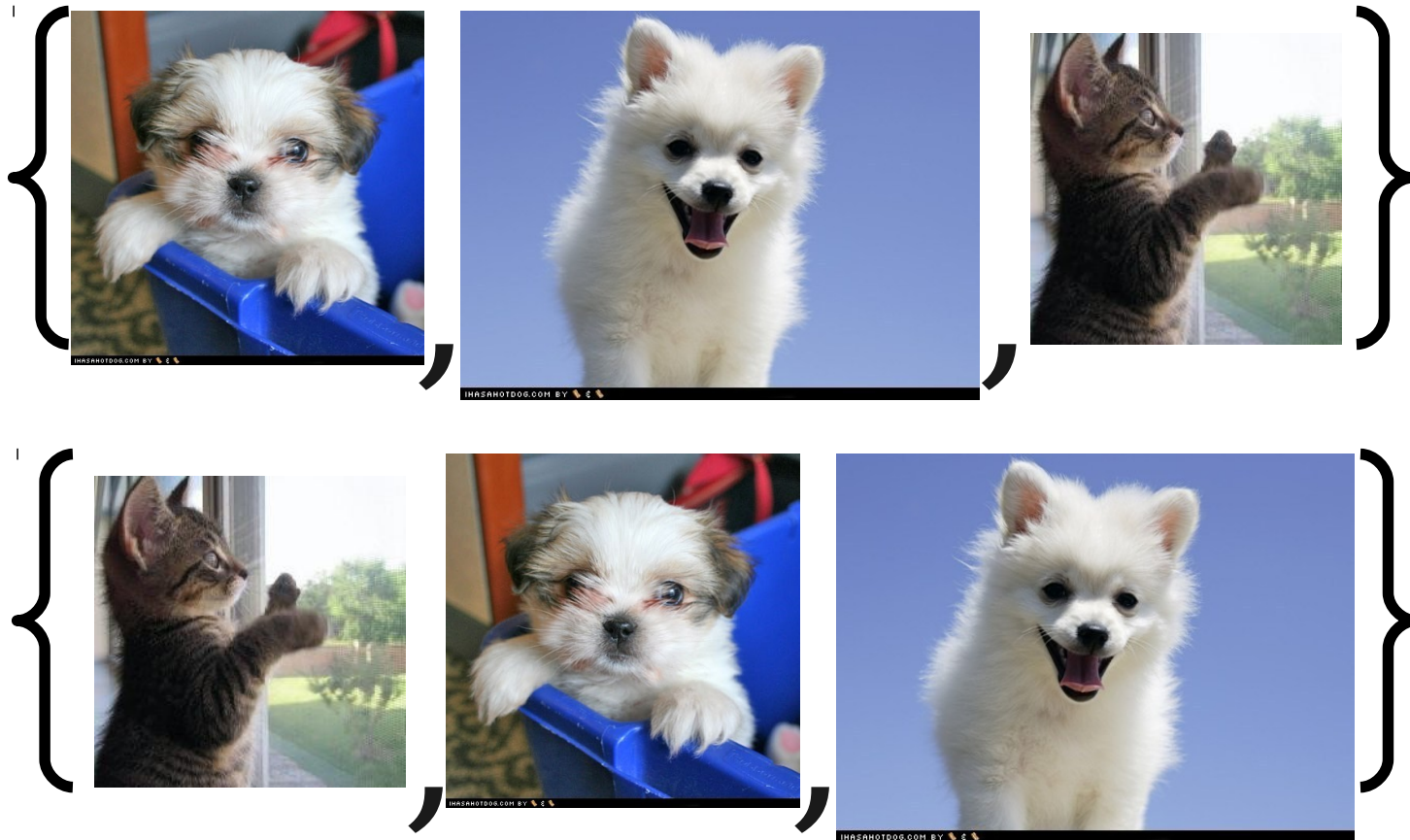
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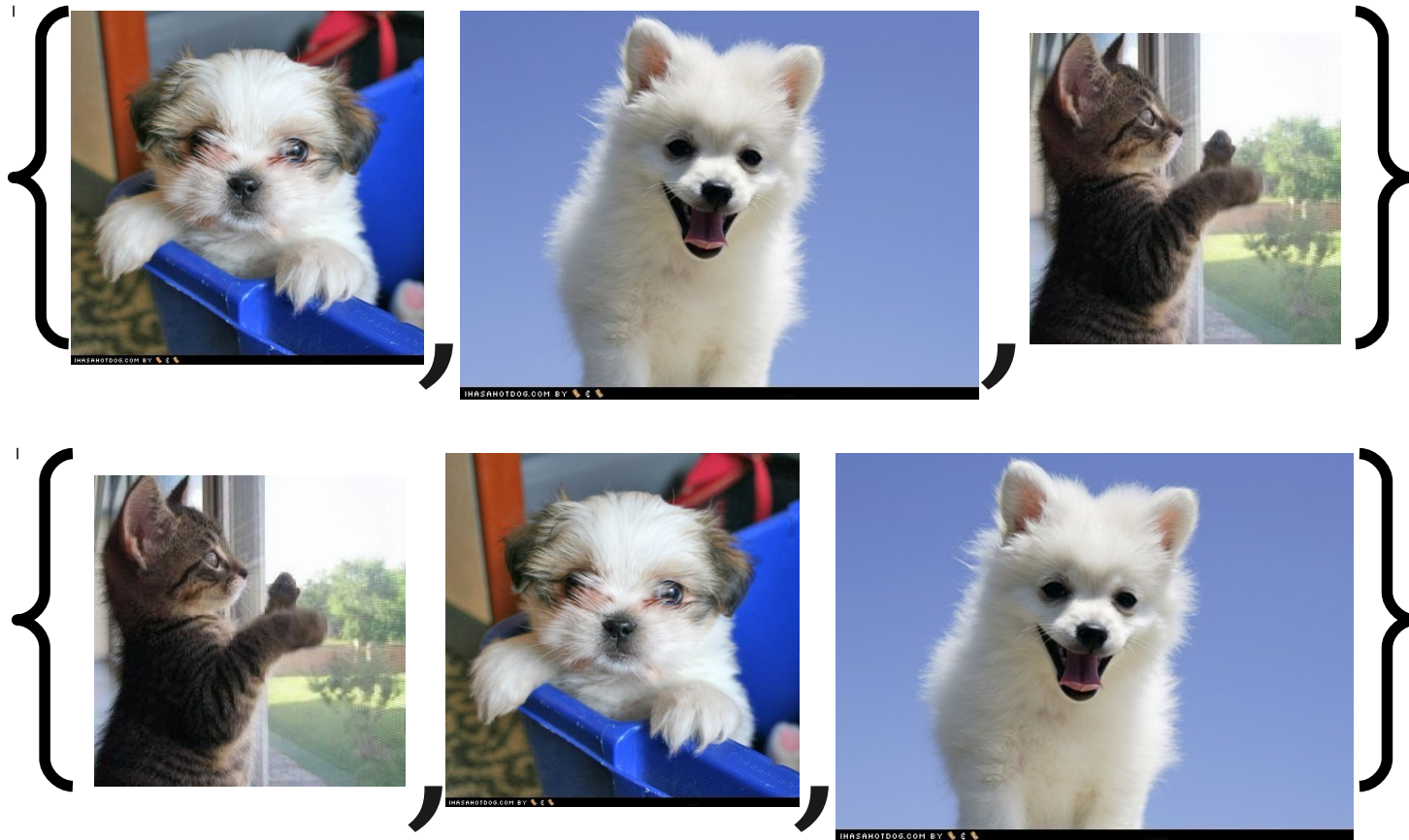
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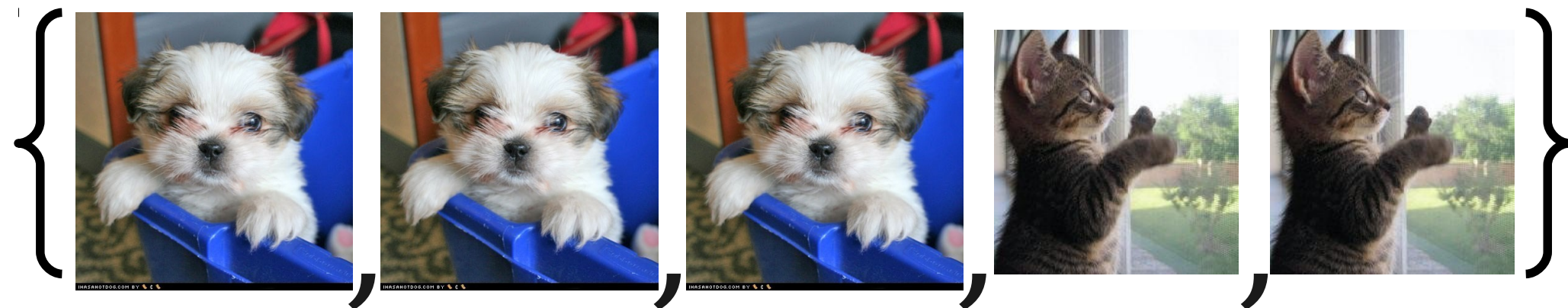
These are
the same
set!

A **set** is an **unordered** collection of distinct objects, which may be anything (including other sets).

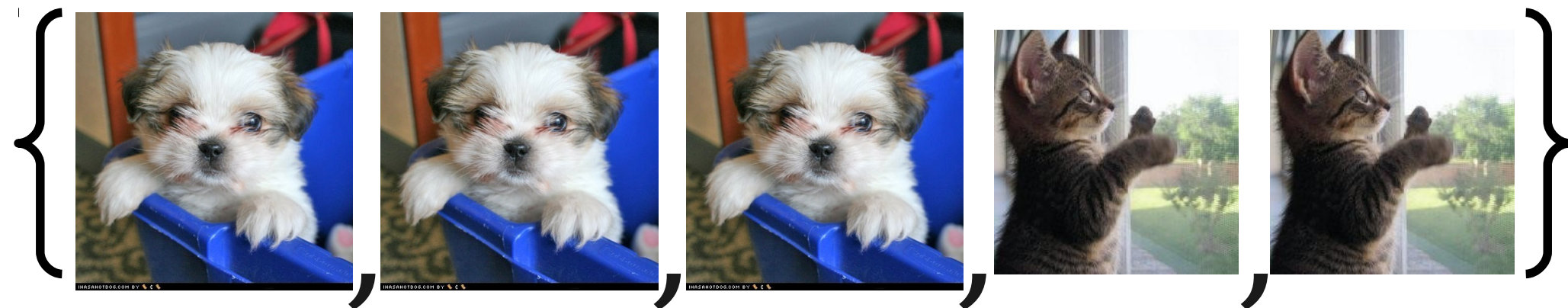
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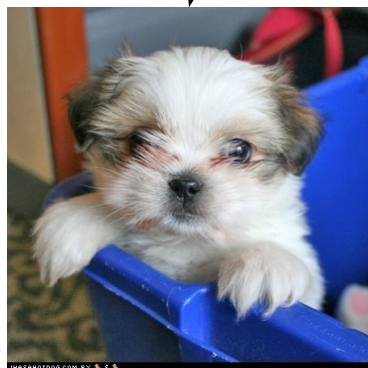
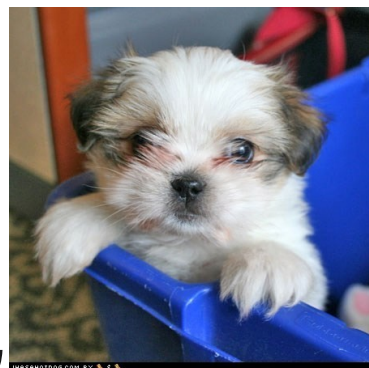
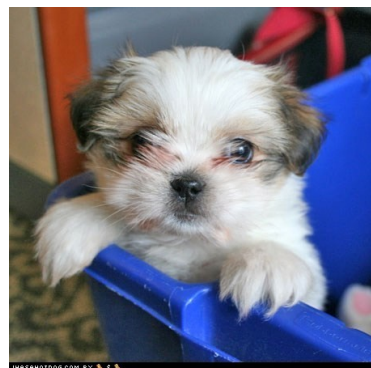
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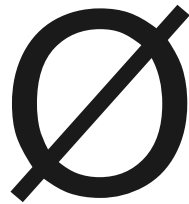
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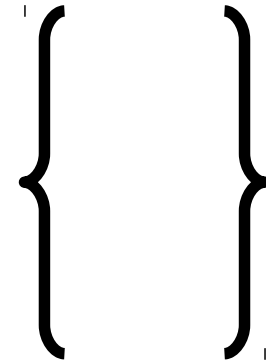
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The empty set
contains no elements.

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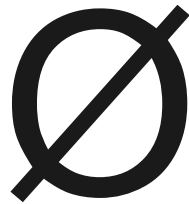


We denote it
with this symbol

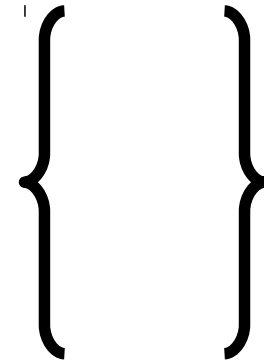


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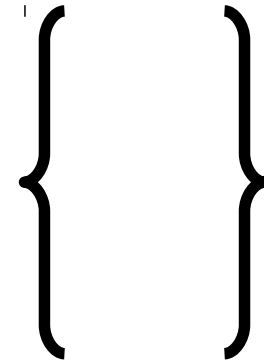
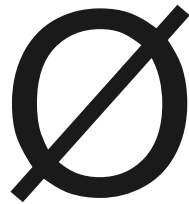


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A **set** is an unordered collection of distinct objects, which may be anything (including other sets).

This symbol means "is defined as"



We denote it
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Membership

Membership



Membership



Is



in this set?

Membership



Is



in this set?

Membership



Membership



Is



in this set?

Set Membership

- Given a set S and an object x , we write

$$x \in S$$

if x is contained in S , and

$$x \notin S$$

otherwise.

- If $x \in S$, we say that x is an **element** of S .
- Given any object and any set, either that object is in the set or it isn't.

Infinite Sets

- Sets can be infinitely large.
- The **natural numbers**, \mathbb{N} : $\{ 0, 1, 2, 3, \dots \}$
 - Some authors (including Sipser) don't include zero; in this class, assume that 0 is a natural number.
- The **integers**, \mathbb{Z} : $\{ \dots, -2, -1, 0, 1, 2, \dots \}$
 - Z is from German “Zahlen.”
- The **real numbers**, \mathbb{R} , including rational and irrational numbers.

Constructing Sets from Other Sets

- Consider these English descriptions:
 - “All even numbers.”
 - “All real numbers less than 137.”
 - “All negative integers.”
- We can't list their (infinitely many!) elements.
- How would we rigorously describe them?

The Set of Even Numbers

$$\{ x \mid x \in \mathbb{N} \text{ and } x \text{ is even} \}$$

The Set of Even Numbers

$$\{ \mathbf{x} \mid x \in \mathbb{N} \text{ and } x \text{ is even} \}$$

The set of all x



The Set of Even Numbers

$$\{ \mathbf{x} \mid x \in \mathbb{N} \text{ and } x \text{ is even} \}$$

The set of all x

where

The Set of Even Numbers

$$\{ x \mid x \in \mathbb{N} \text{ and } x \text{ is even} \}$$

The set of all x

where

x is in the set of
natural numbers

The Set of Even Numbers

$$\{ \textcolor{olive}{x} \mid \textcolor{violet}{x} \in \mathbb{N} \text{ and } \textcolor{teal}{x} \text{ is even} \}$$

The set of all x

where

x is in the set of
natural numbers

and x is even

Set Builder Notation

- A set may be specified in **set-builder notation**:

$$\{ x \mid \textit{some property } x \textit{ satisfies} \}$$

- For example:

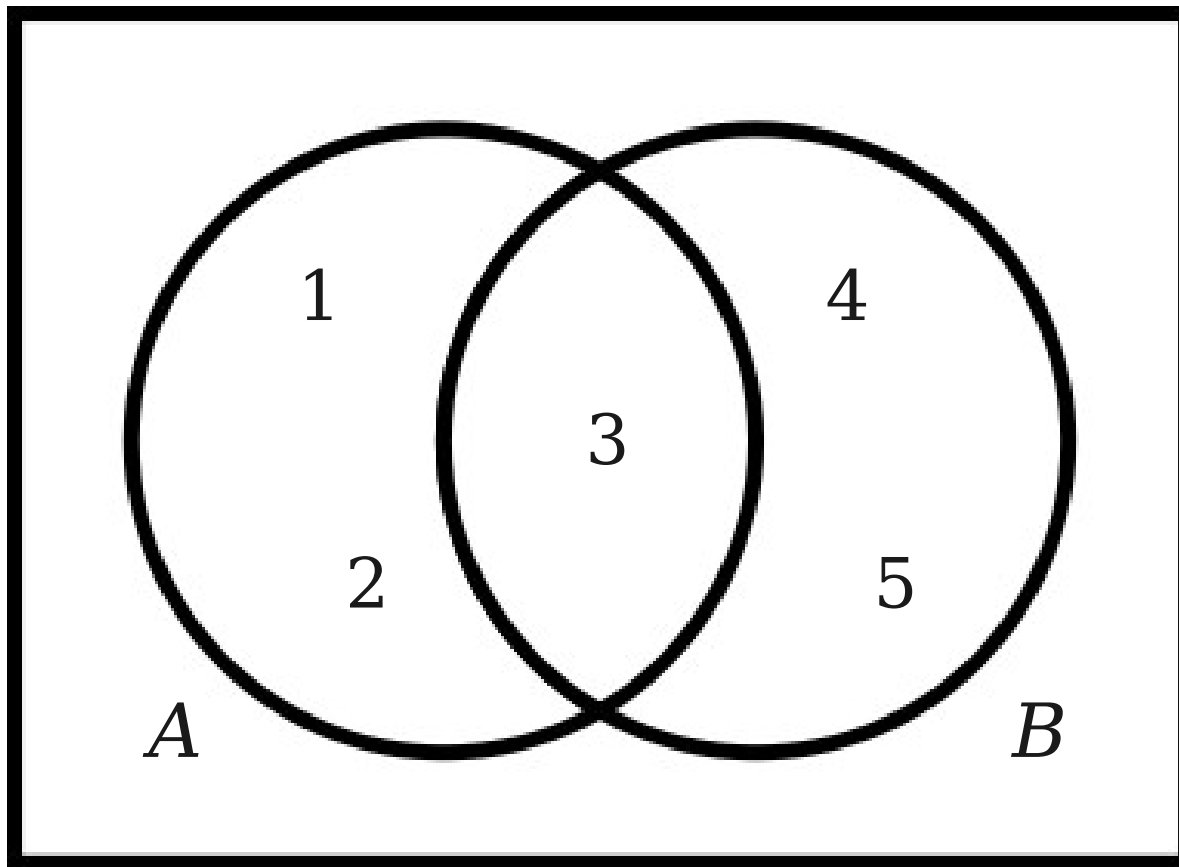
$$\{ r \mid r \in \mathbb{R}, r < 137 \}$$

$$\{ n \mid n \text{ is a perfect square} \}$$

$$\{ x \mid x \text{ is a set of US currency} \}$$

Combining Sets

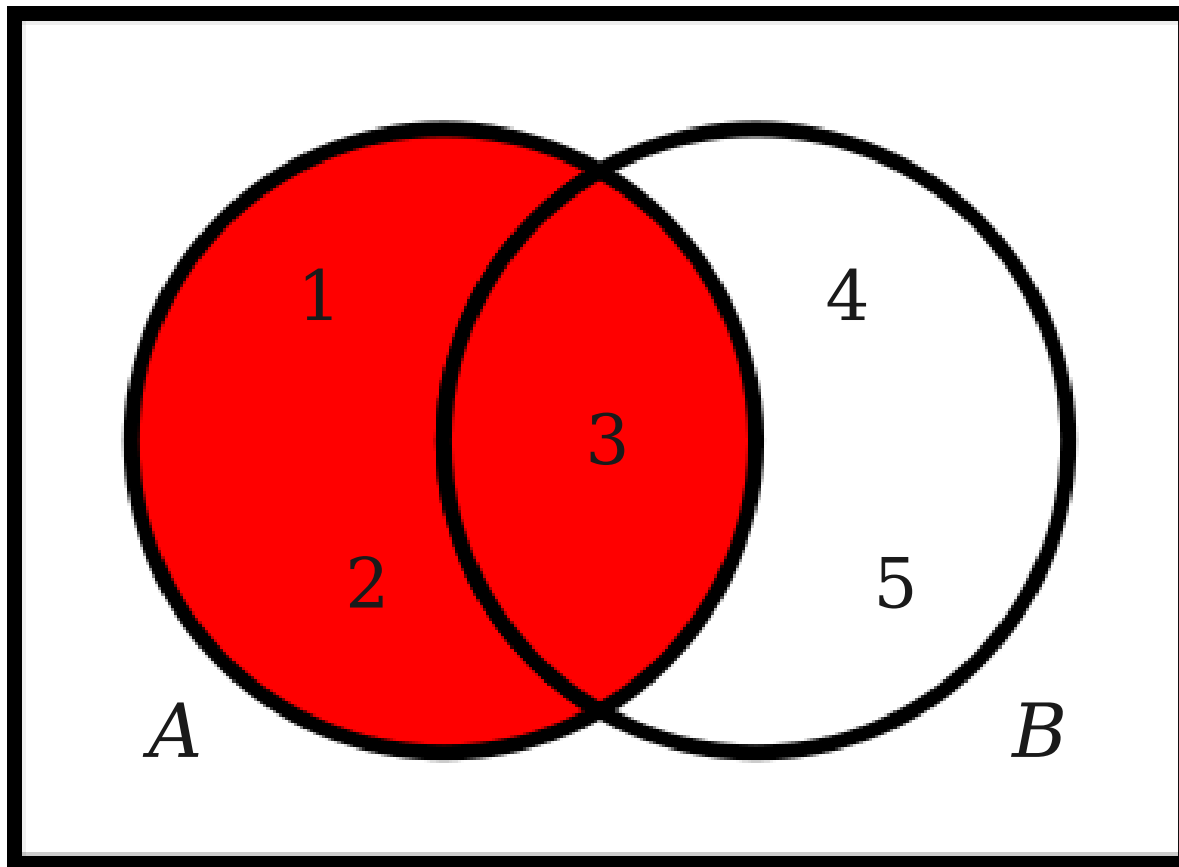
Venn Diagrams



$$A = \{ 1, 2, 3 \}$$

$$B = \{ 3, 4, 5 \}$$

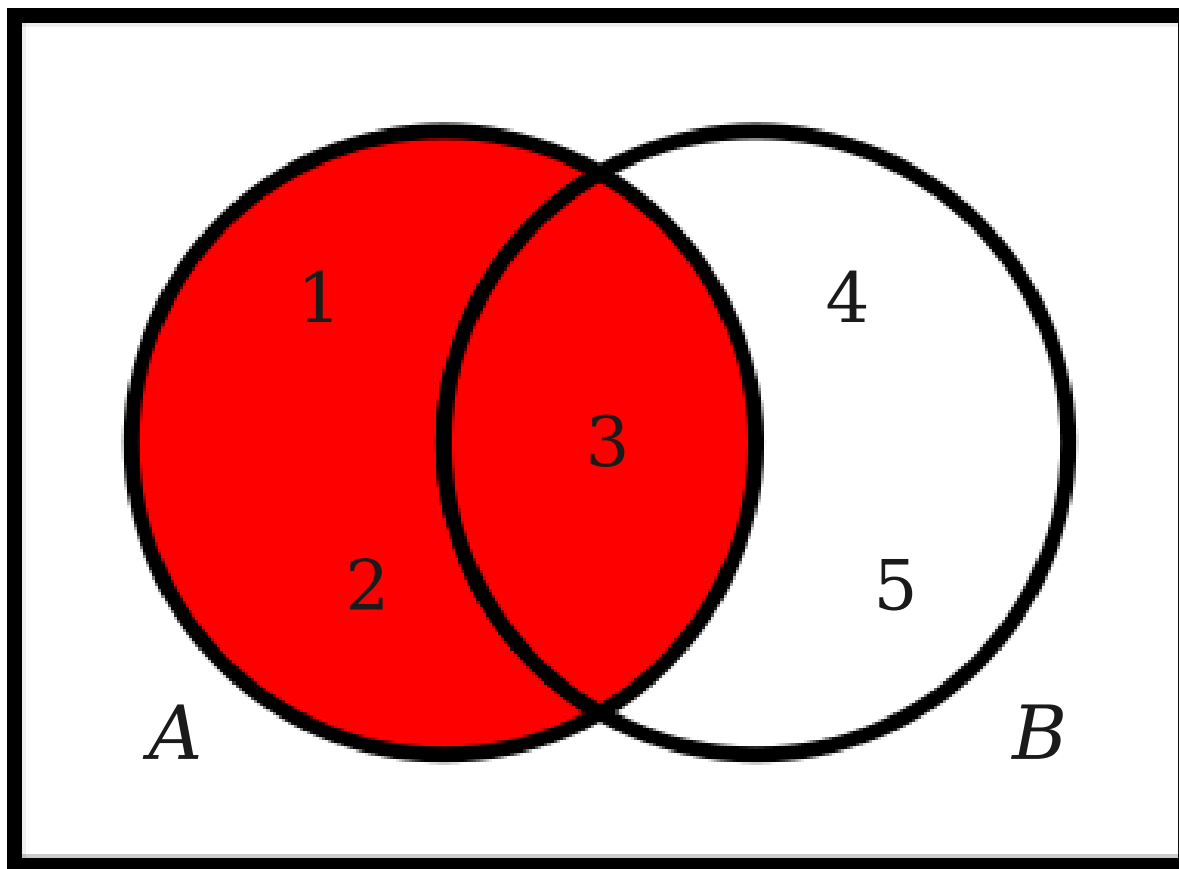
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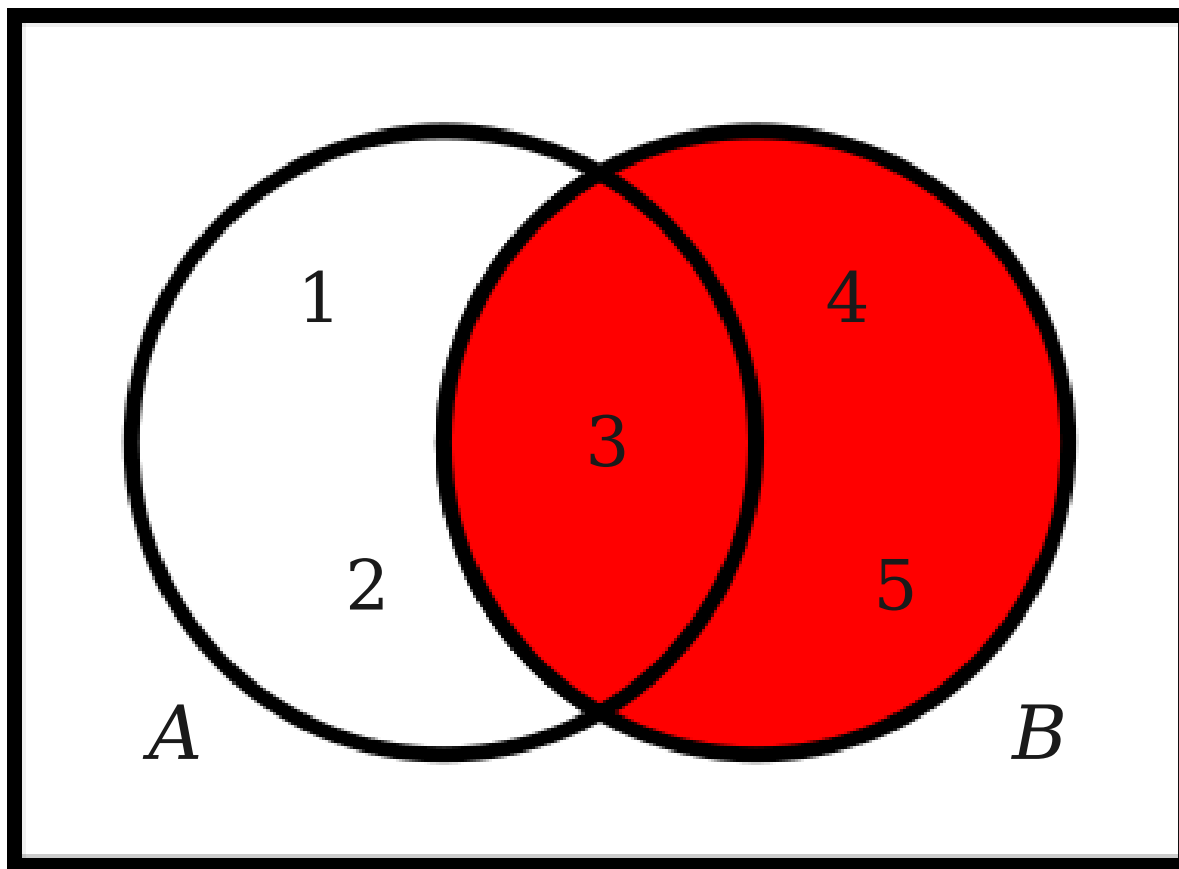


A

$$A = \{ 1, 2, 3 \}$$

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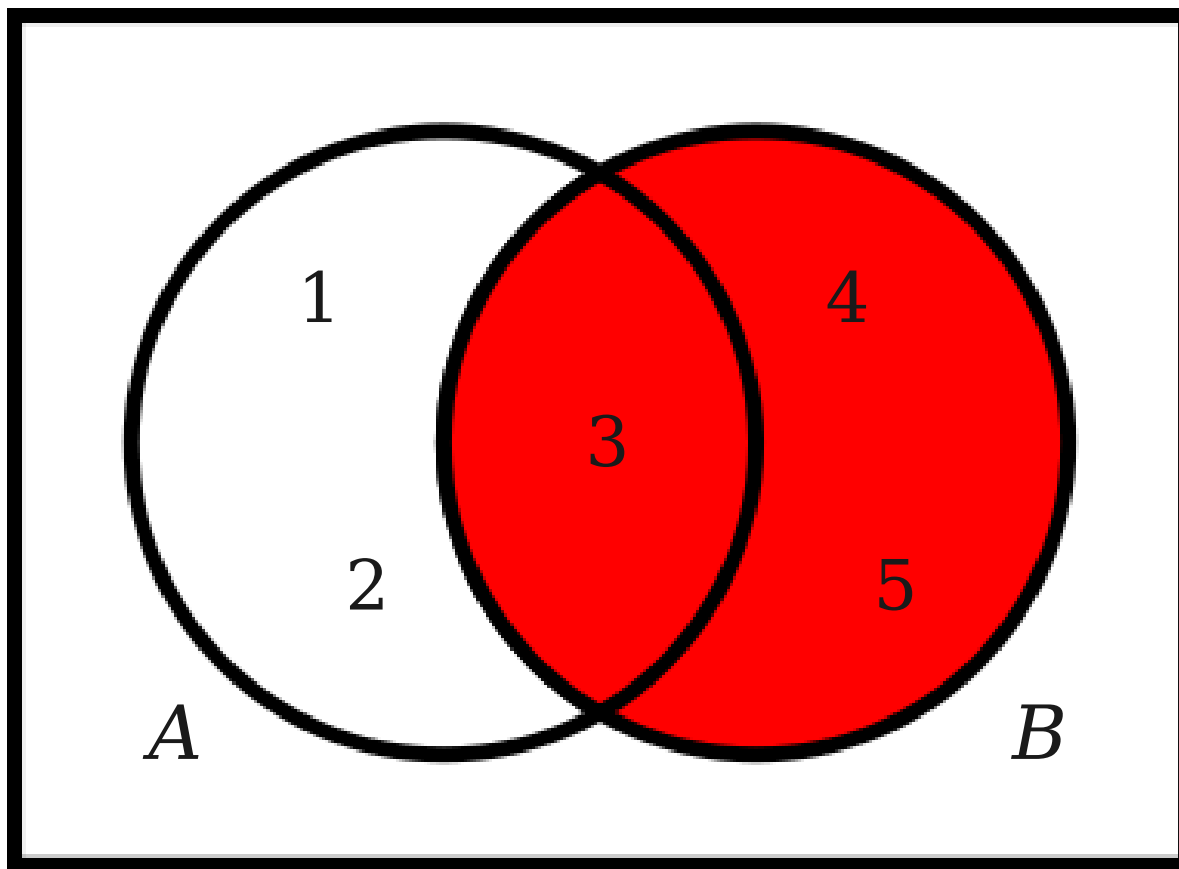
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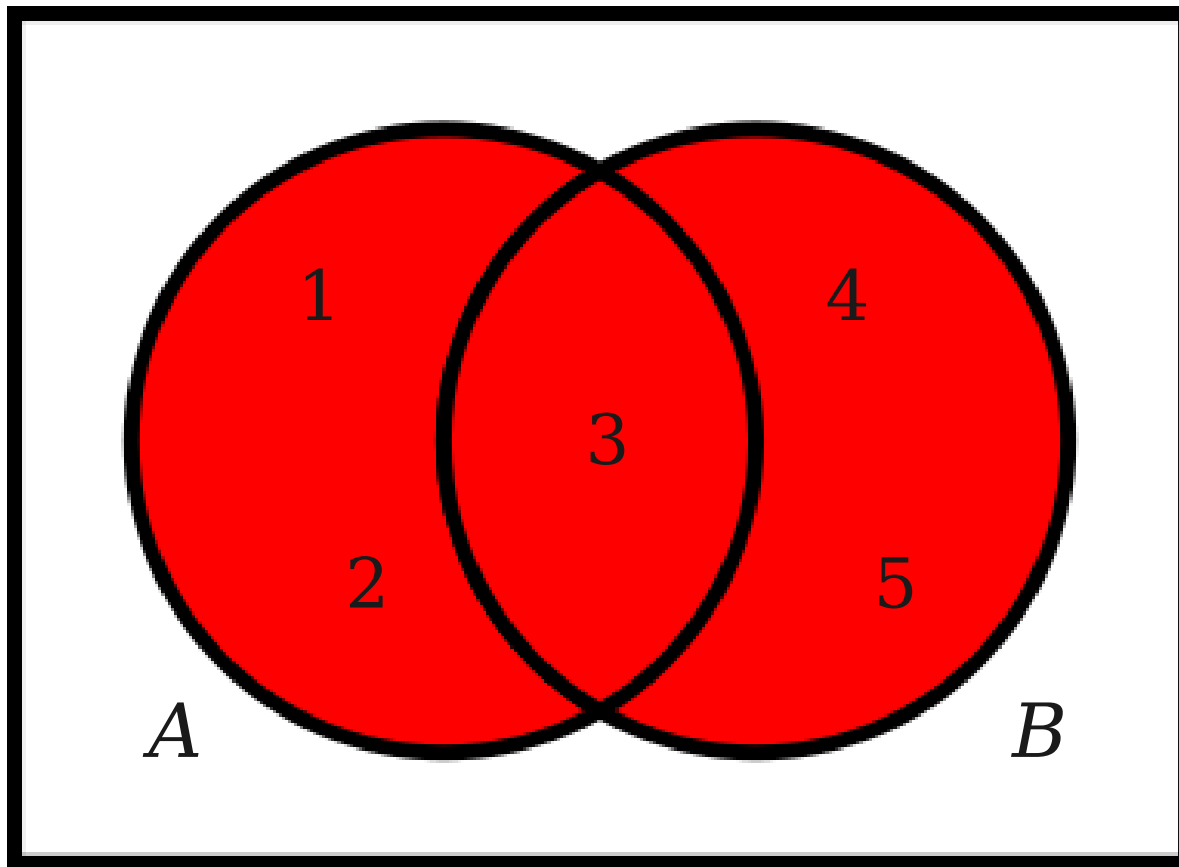


B

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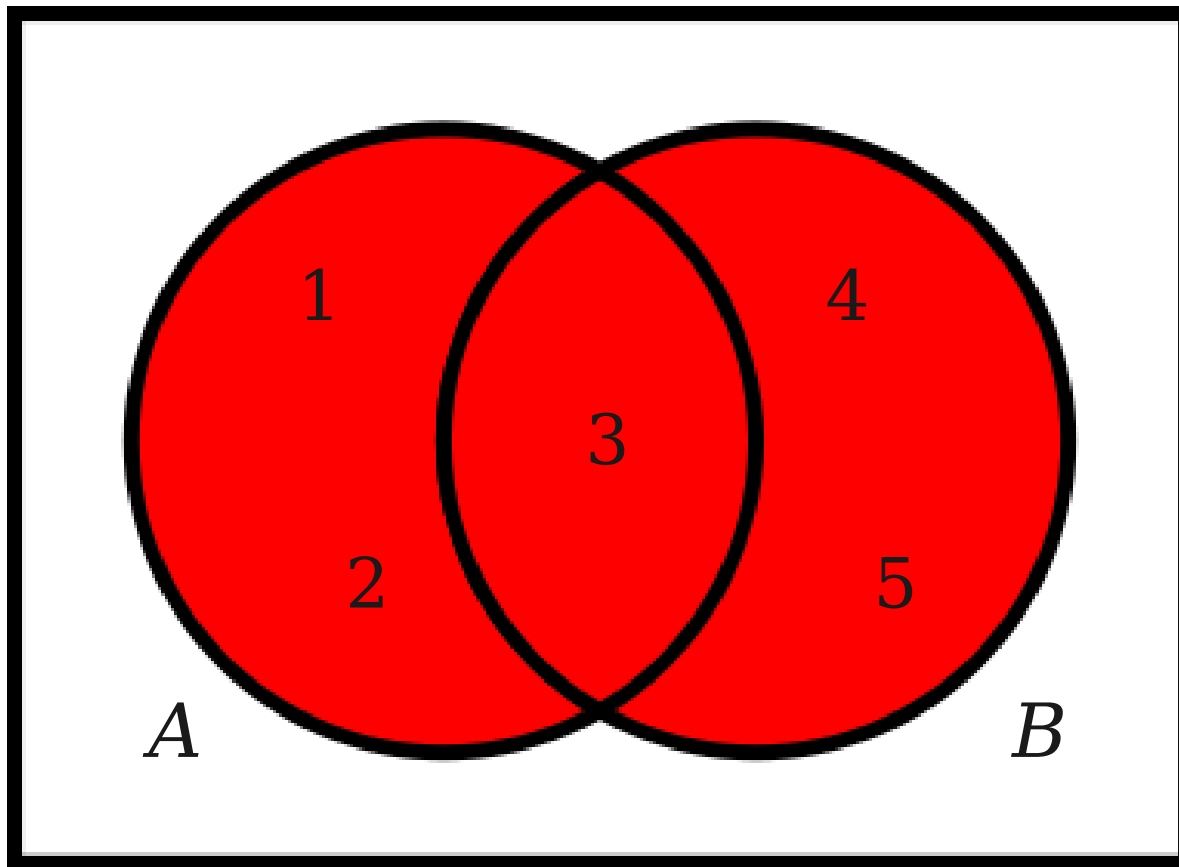
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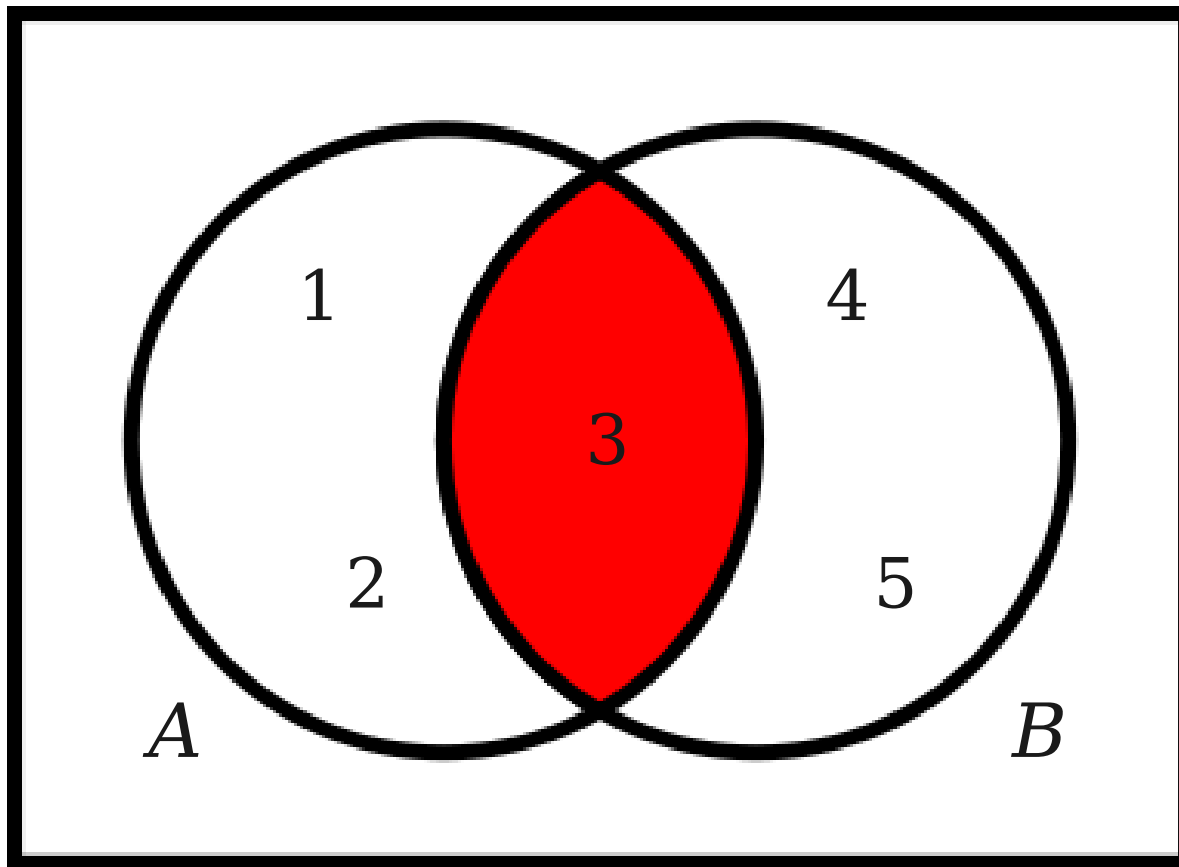


Union
 $A \cup B$
 $\{ 1, 2, 3, 4, 5 \}$

$$A = \{ 1, 2, 3 \}$$

$$B = \{ 3, 4, 5 \}$$

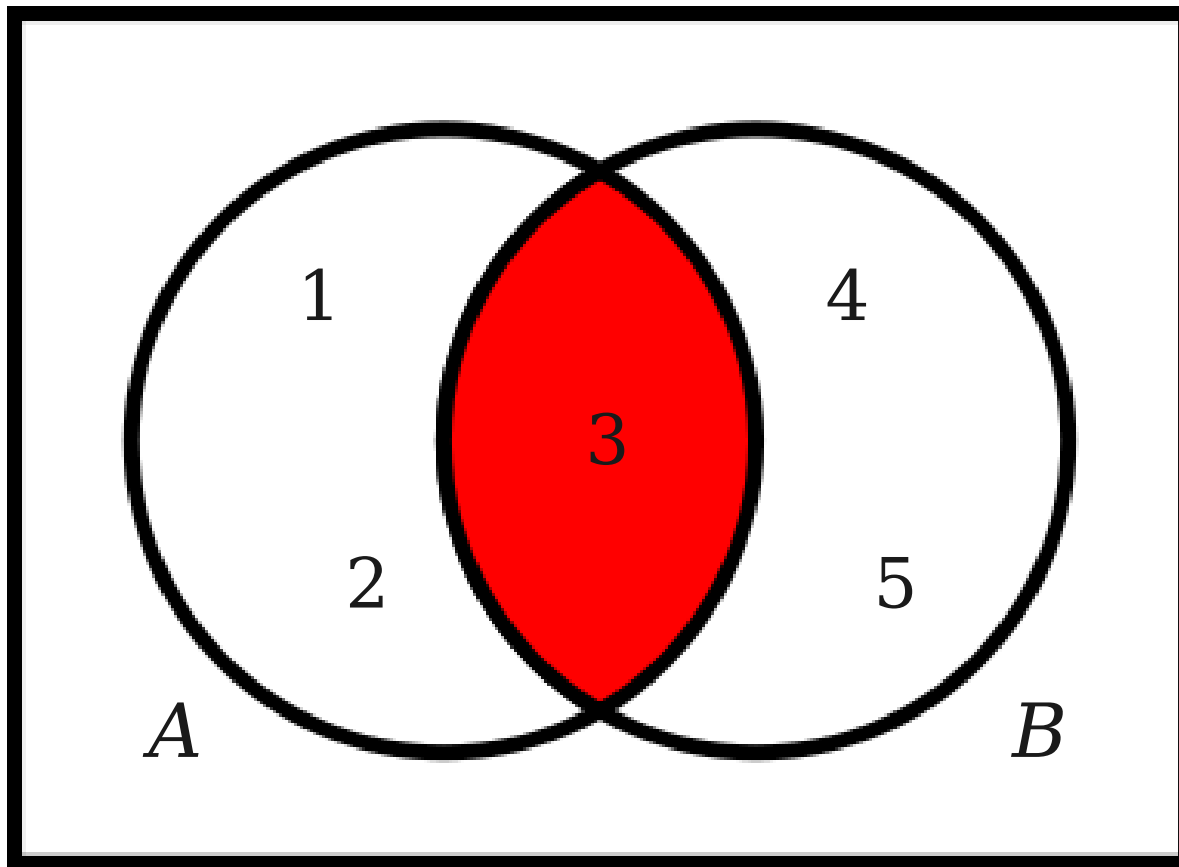
Venn Diagrams



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Venn Diagrams



Intersection

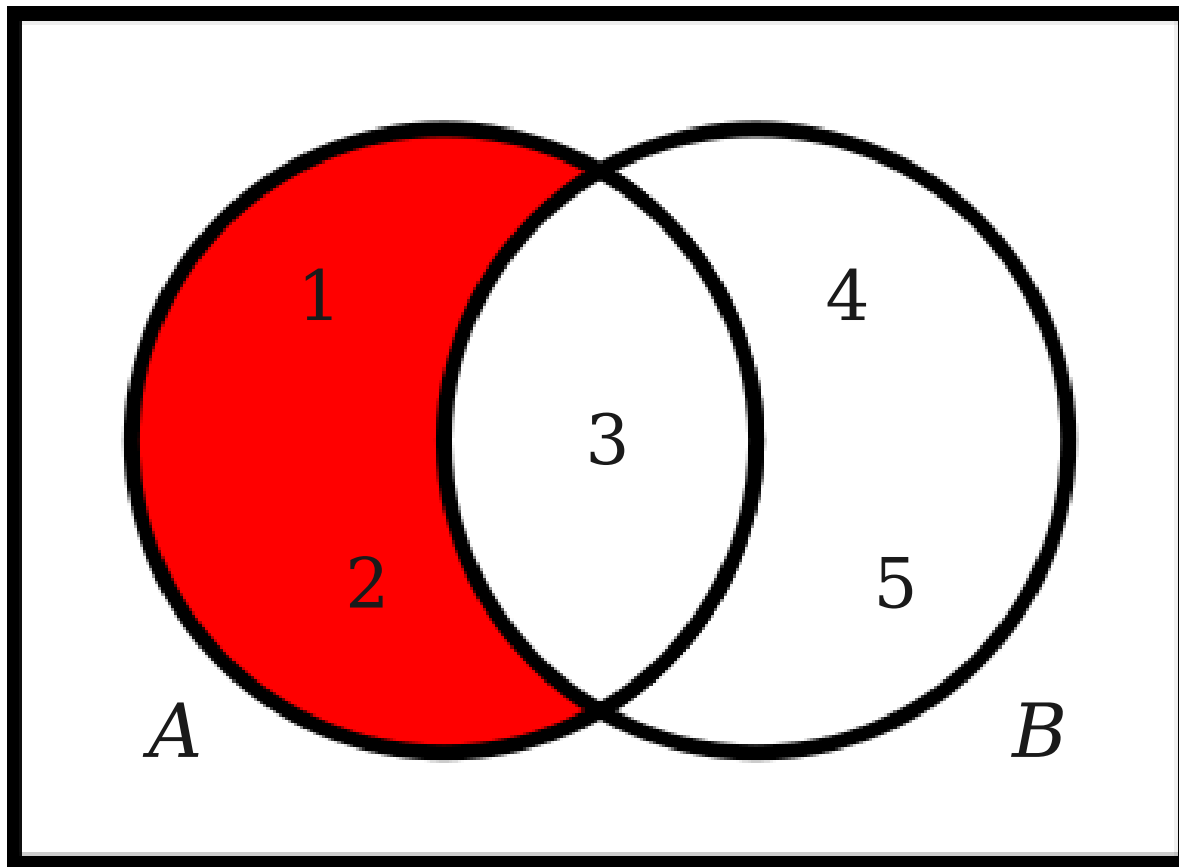
$$A \cap B$$

$$\{ 3 \}$$

$$A = \{ 1, 2, 3 \}$$

$$B = \{ 3, 4, 5 \}$$

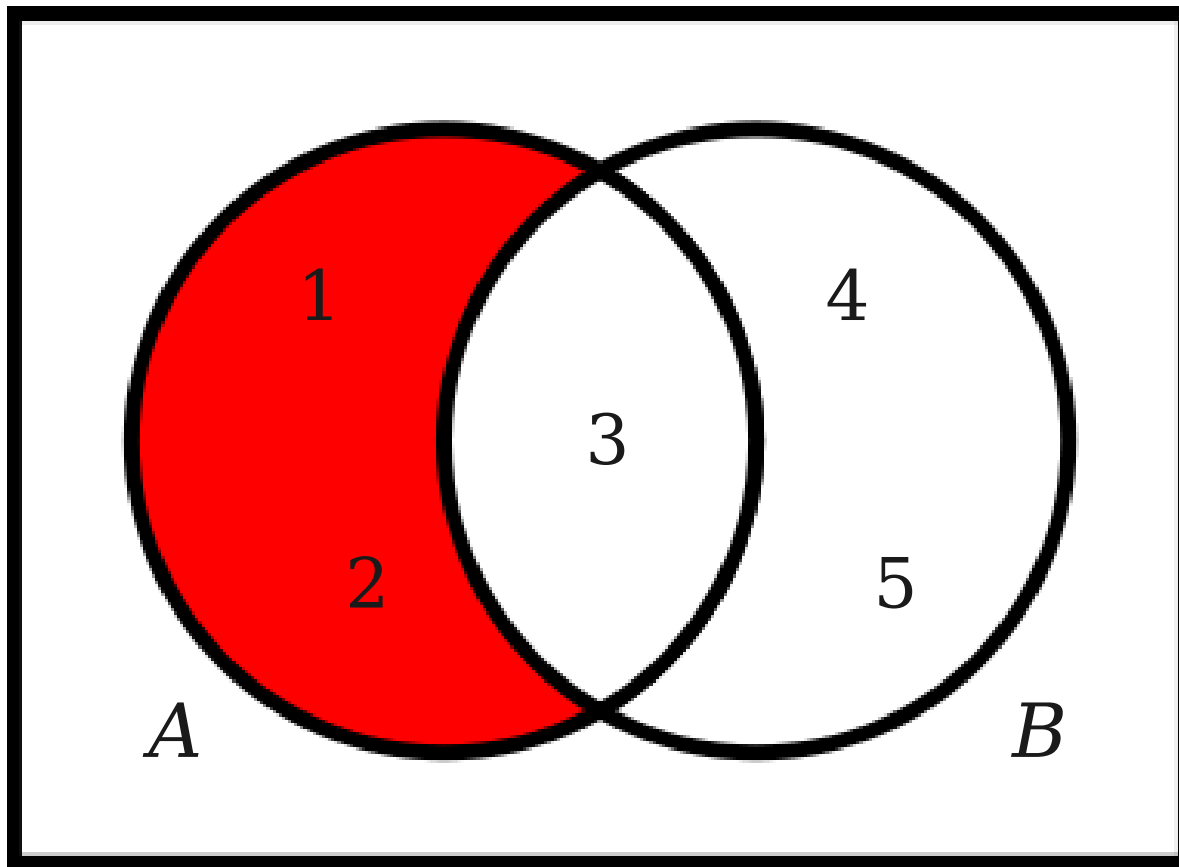
Venn Diagrams



$$A = \{ 1, 2, 3 \}$$

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Venn Diagrams



Difference

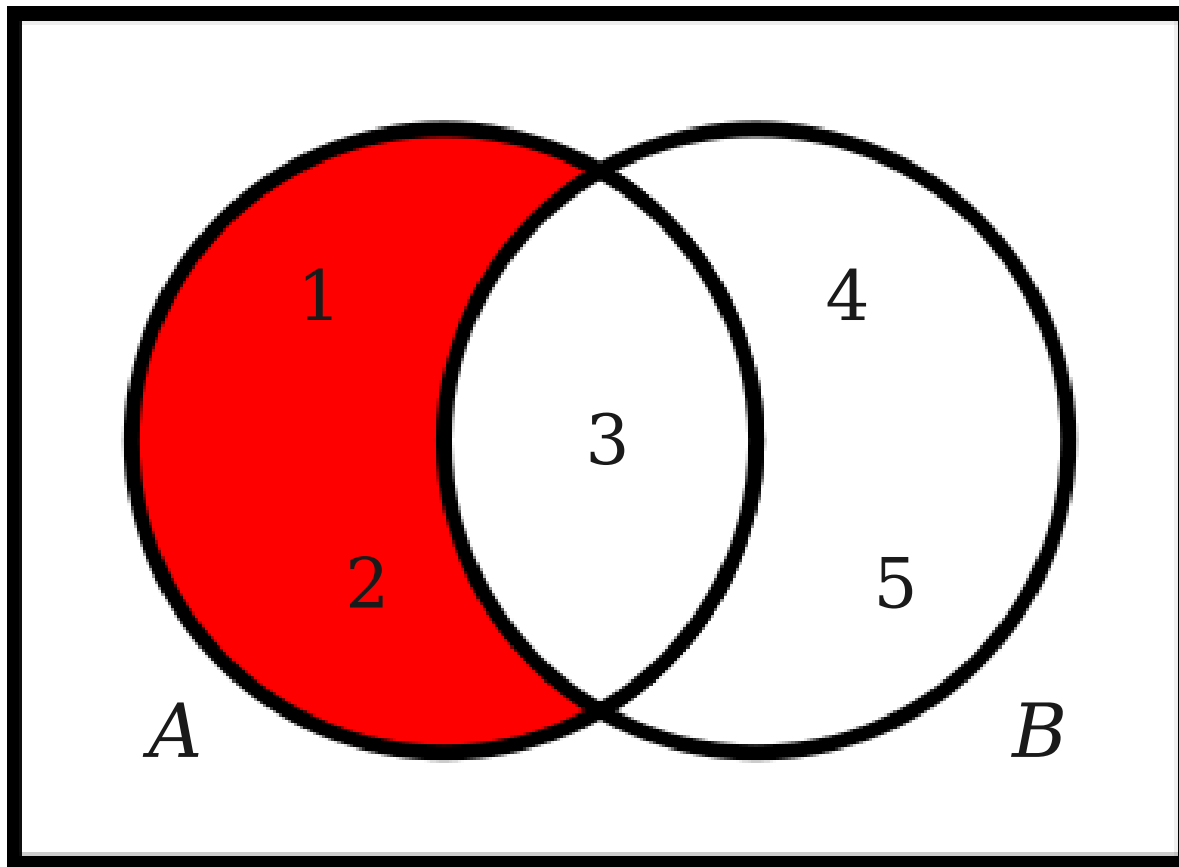
$$A - B$$

$$\{ 1, 2 \}$$

$$A = \{ 1, 2, 3 \}$$

$$B = \{ 3, 4, 5 \}$$

Venn Diagrams



Difference

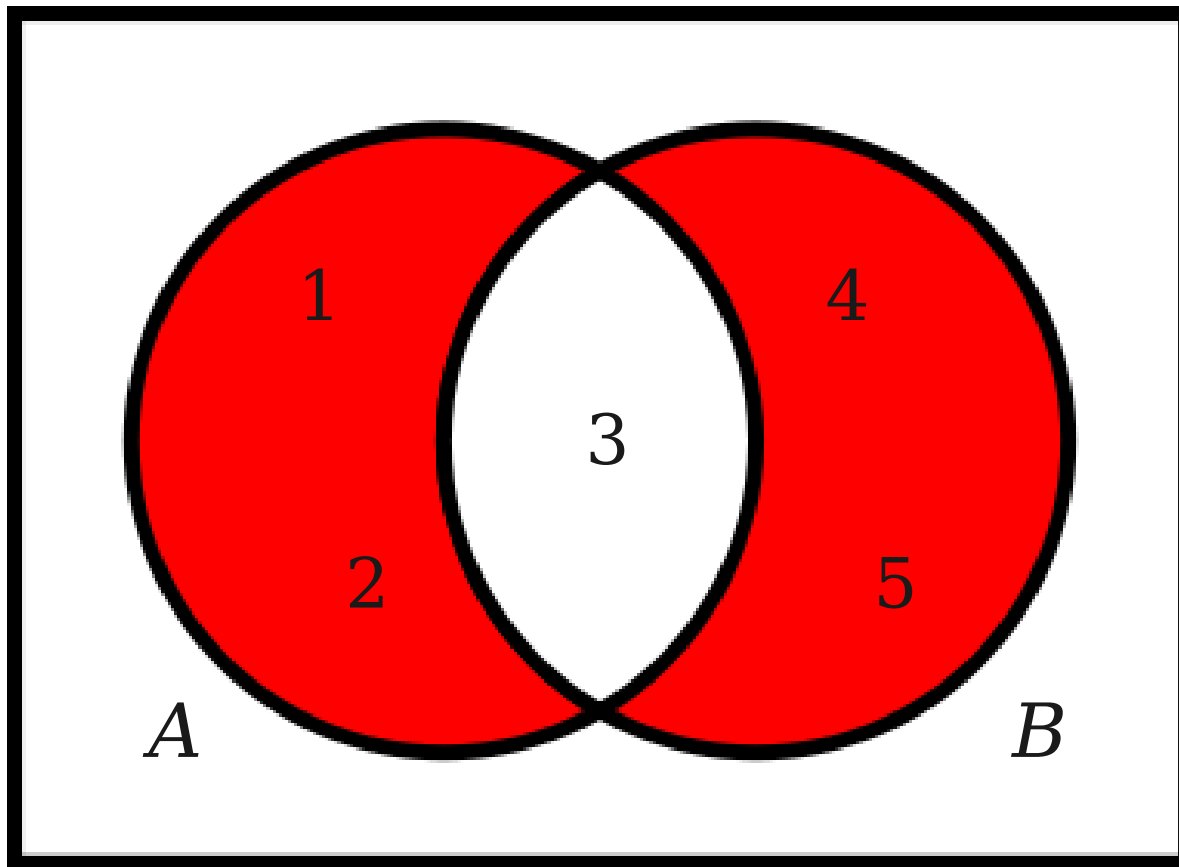
$$A \setminus B$$

$$\{ 1, 2 \}$$

$$A = \{ 1, 2, 3 \}$$

$$B = \{ 3, 4, 5 \}$$

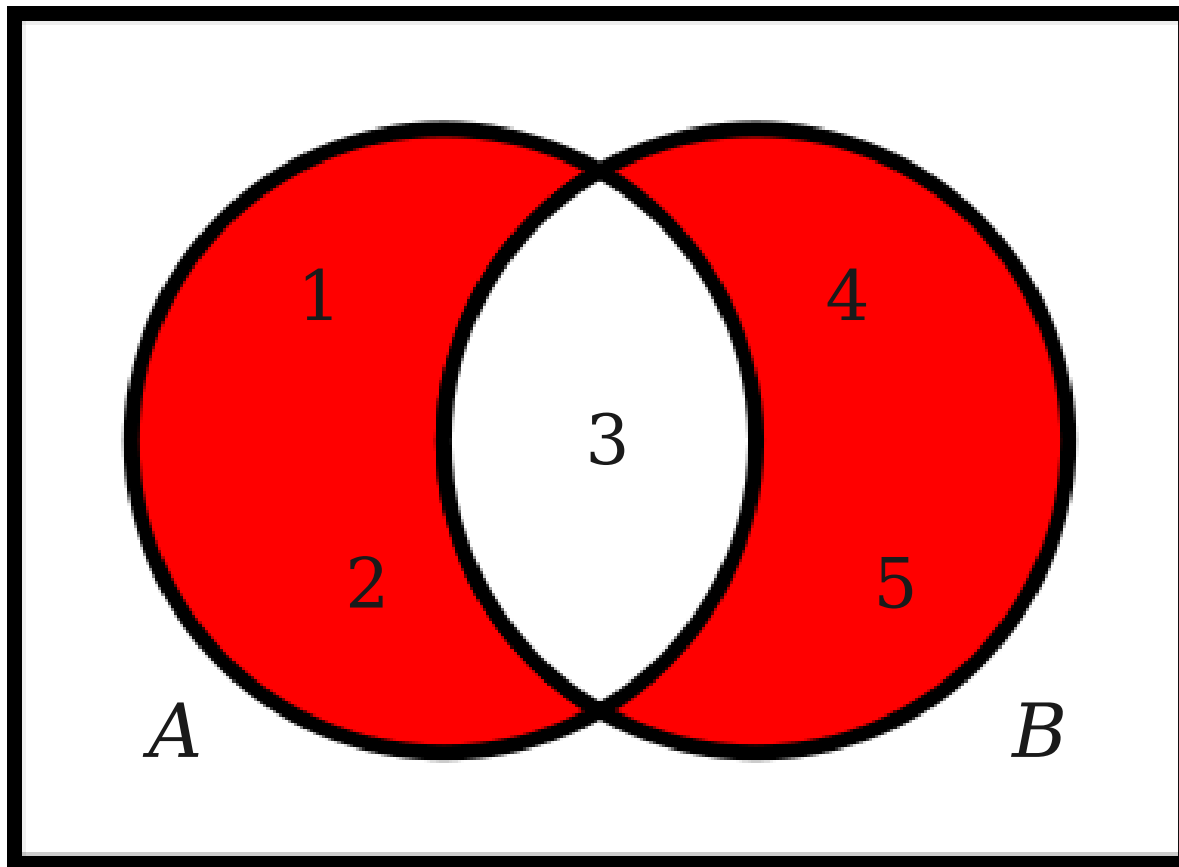
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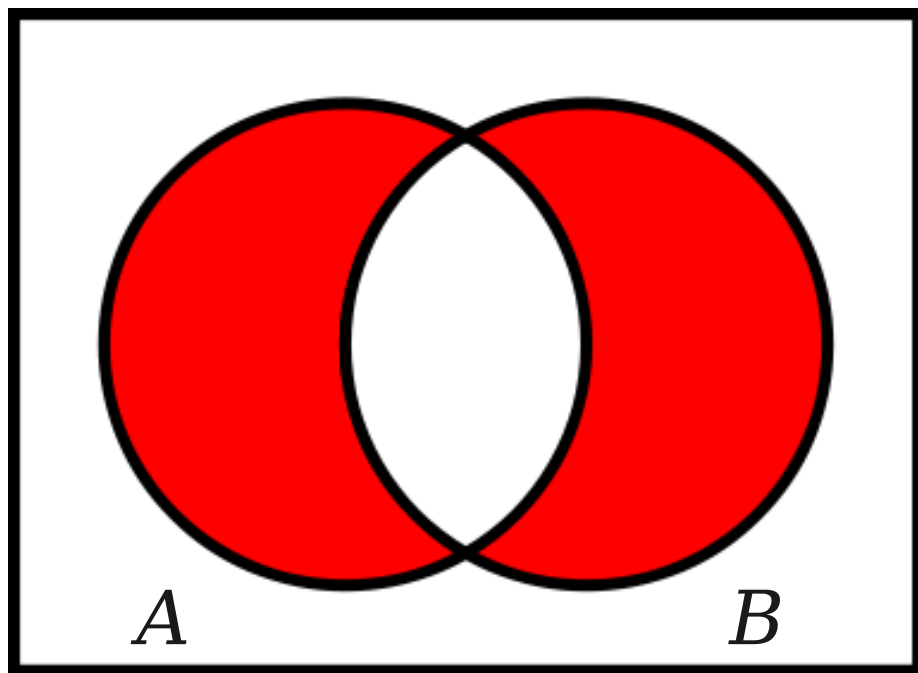
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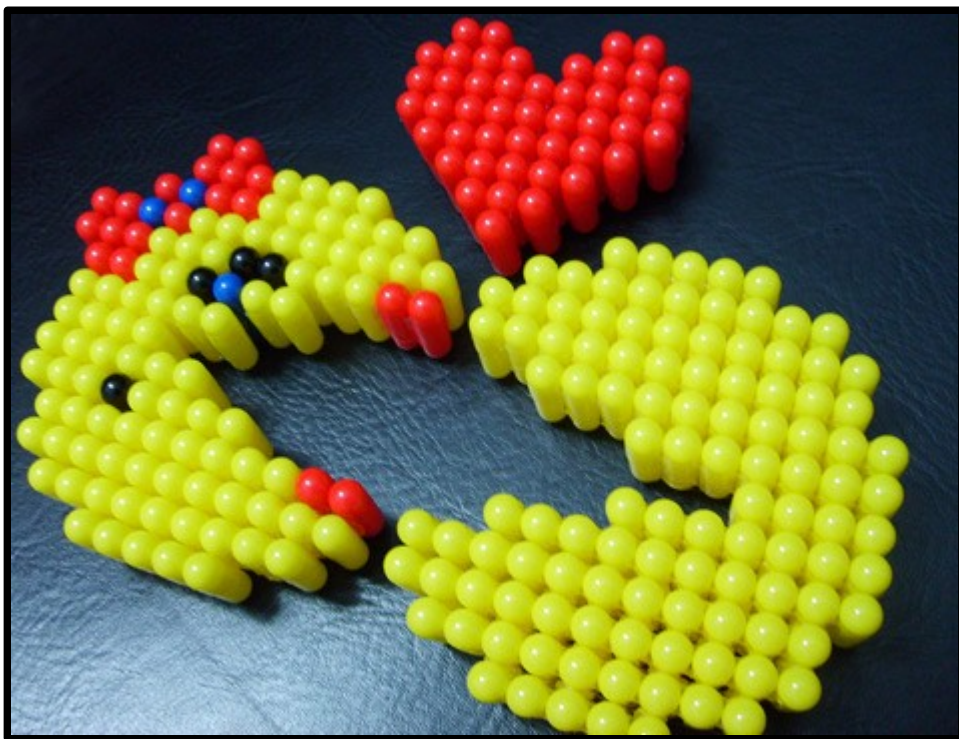
Symmetric
Difference
 $A \Delta B$
 $\{ 1, 2, 4, 5 \}$

$$A = \{ 1, 2, 3 \}$$
$$B = \{ 3, 4, 5 \}$$

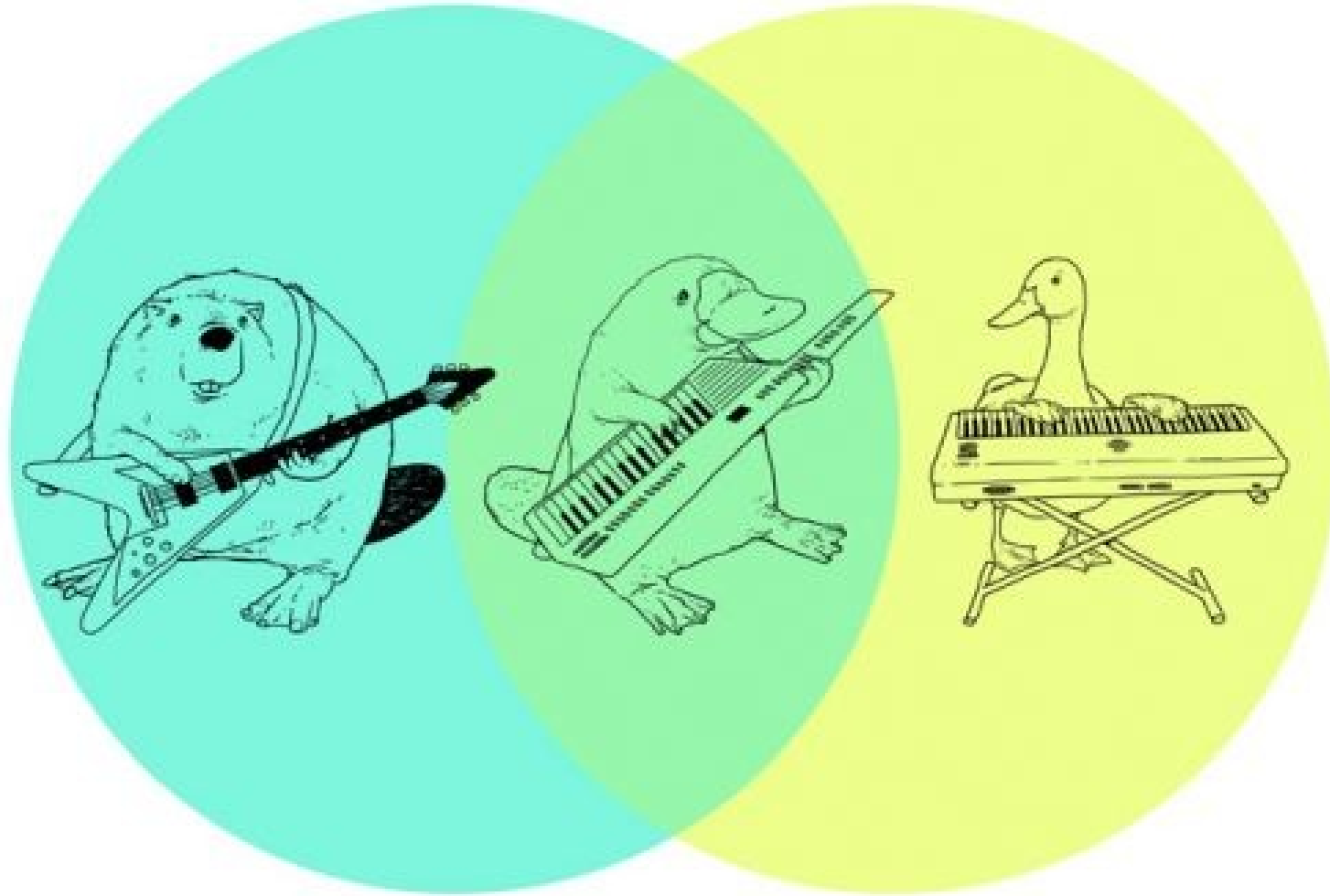
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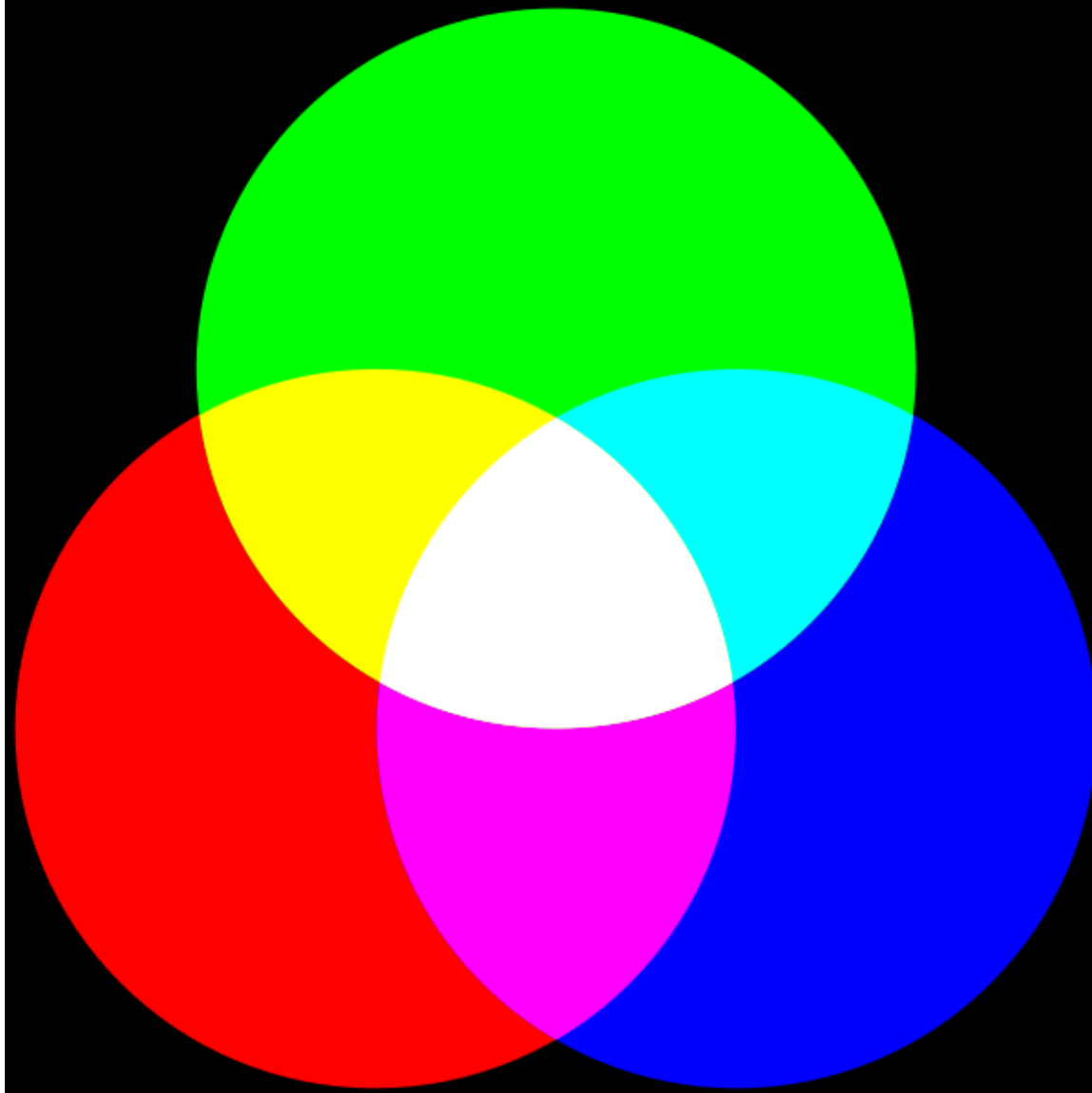
$$A \Delta B$$



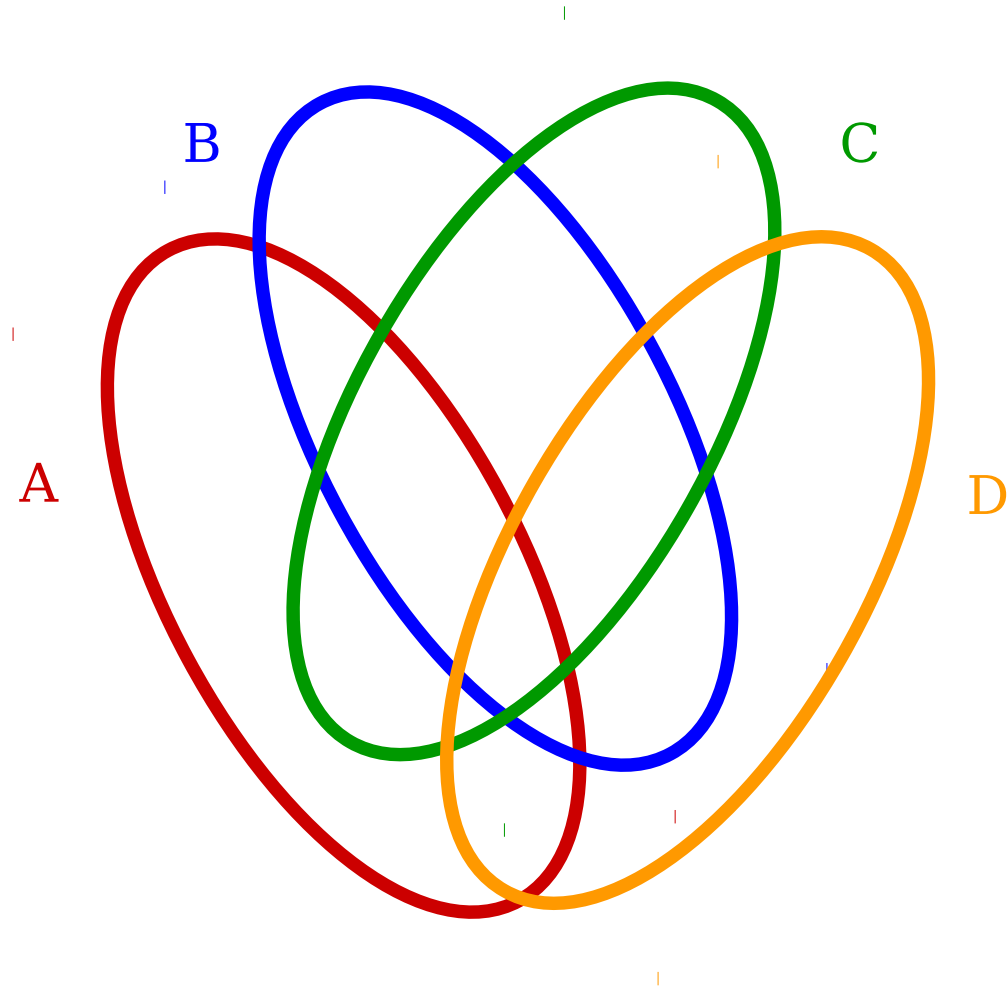
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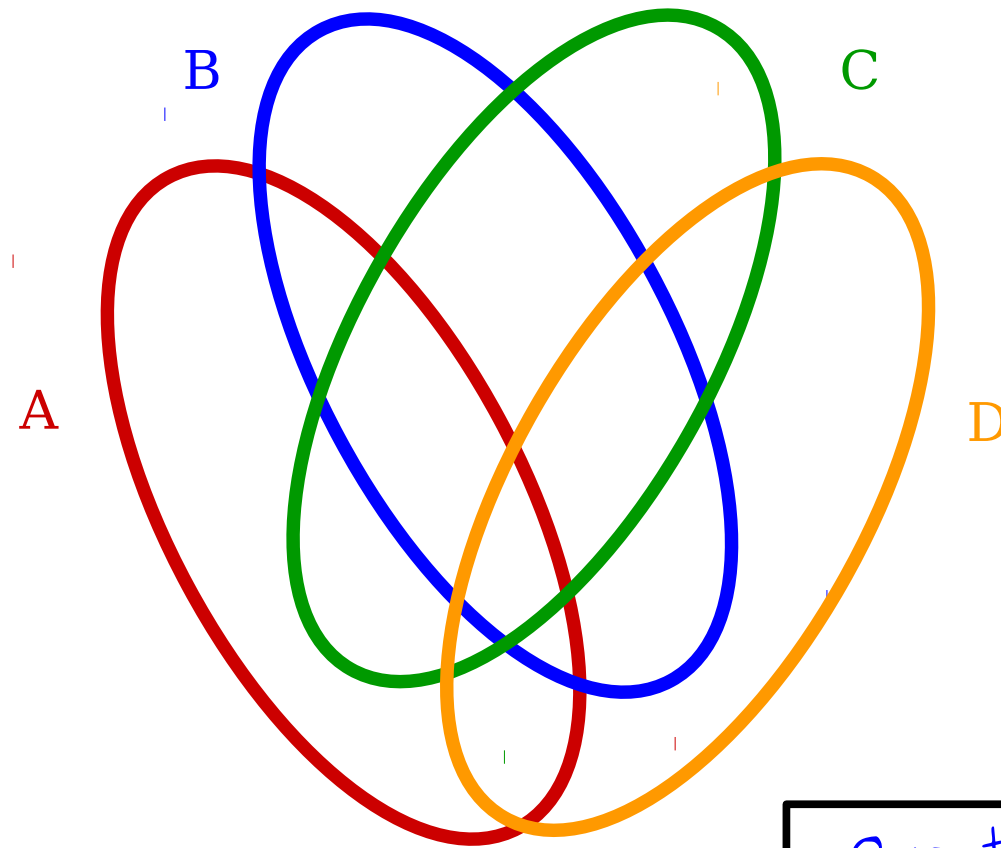
Venn Diagrams for Three Sets



Venn Diagrams for Four Sets



Venn Diagrams for Four Sets



Question to ponder:
why can't we just
draw four circles?

A Fun Website:
Venn Diagrams for Seven Sets

<http://moebio.com/research/sevensets/>

Subsets and Power Sets

Subsets

- A set S is a **subset** of some set T if every element of S is also an element in T :

If $x \in S$, then $x \in T$.

- We denote this as **$S \subseteq T$** .
- Examples:
 - $\{ 1, 2, 3 \} \subseteq \{ 1, 2, 3, 4 \}$
 - $\mathbb{N} \subseteq \mathbb{Z}$ (*every natural number is an integer*)
 - $\mathbb{Z} \subseteq \mathbb{R}$ (*every integer is a real number*)

What About the Empty Set?

- A set S is a **subset** of some set T if every element of S is also an element in T :

If $x \in S$, then $x \in T$.

- Is $\emptyset \subseteq S$ for any set S ?
- **Yes:** The above statement is true.
- **Vacuous truth:** A statement that is true because it does not apply to anything.
 - “All unicorns are blue.”
 - “All unicorns are pink.”

Proper Subsets

- By definition, any set is a subset of itself.
(*Why?*)
- A **proper subset** of a set S is a set T such that
 - $T \subseteq S$
 - $T \neq S$
- There are multiple notations for this; they all mean the same thing:
 - $T \subsetneq S$
 - $T \subset S$

$$S = \{ \text{Lincoln Penny}, \text{Lincoln Dime} \}$$

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$$\emptyset \{ \text{Lincoln Dime} \} \{ \text{Lincoln Penny} \} \{ \text{Lincoln Penny}, \text{Lincoln Dime} \}$$

$$S = \{ \text{Lincoln Penny}, \text{Kennedy Half Dollar} \}$$

$$\{ \emptyset, \{ \text{Kennedy Half Dollar} \}, \{ \text{Lincoln Penny} \}, \{ \text{Lincoln Penny}, \text{Kennedy Half Dollar} \} \}$$

$$S = \left\{ \text{Lincoln Penny}, \text{Kennedy Half Dollar} \right\}$$

$$\mathcal{P}(S) = \left\{ \emptyset, \left\{ \text{Kennedy Half Dollar} \right\}, \left\{ \text{Lincoln Penny} \right\}, \left\{ \text{Lincoln Penny}, \text{Kennedy Half Dollar} \right\} \right\}$$

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$\mathcal{P}(S)$ is the
power set of S
 (the set of all
 subsets of S)

Cardinalities

Cardinalities

Cardinality

- The **cardinality** of a set is the number of elements it contains.
- We denote it $|S|$.
- Examples:
 - $|\{a, b, c, d, e\}| = 5$
 - $|\{\{a, b\}, \{c, d, e, f, g\}, \{h\}\}| = 3$
 - $|\{1, 2, 3, 3, 3, 3, 3\}| = 3$
 - $|\{x \mid x \in \mathbb{N} \text{ and } x < 137\}| = 137$

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The Cardinality of \mathbb{N}

- What is $|\mathbb{N}|$?
 - There are infinitely many natural numbers.
 - $|\mathbb{N}|$ can't be a natural number, since it's infinitely large.

The Cardinality of \mathbb{N}

- What is $|\mathbb{N}|$?
 - There are infinitely many natural numbers.
 - $|\mathbb{N}|$ can't be a natural number, since it's infinitely large.
- We need to introduce a new term.
- Definition: $|\mathbb{N}| = \aleph_0$
 - Pronounced “Aleph-Zero,” “Aleph-Nought,” or “Aleph-Null”

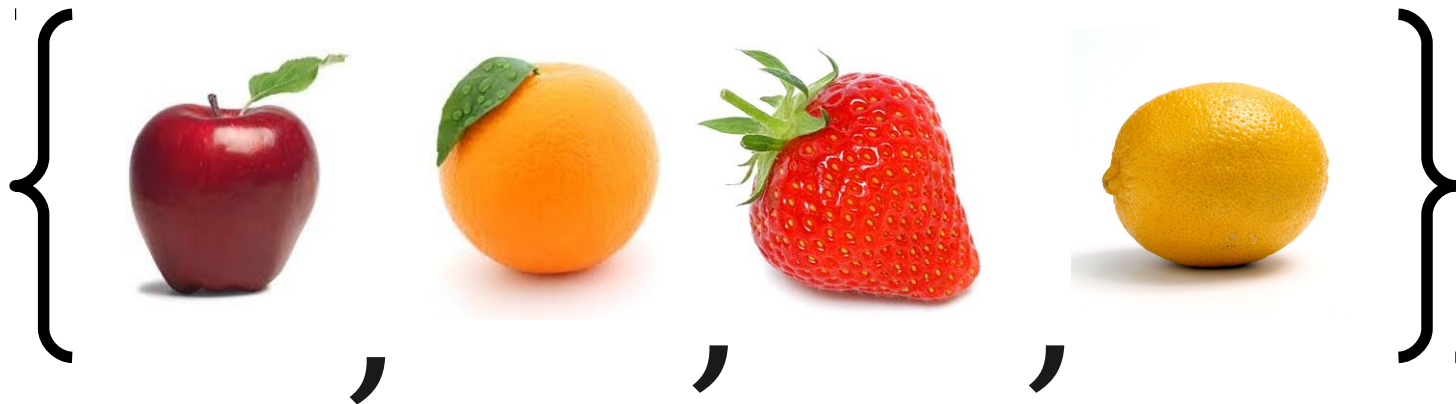
Consider the set

$$S = \{ x \mid x \in \mathbb{N} \text{ and } x \text{ is even} \}$$

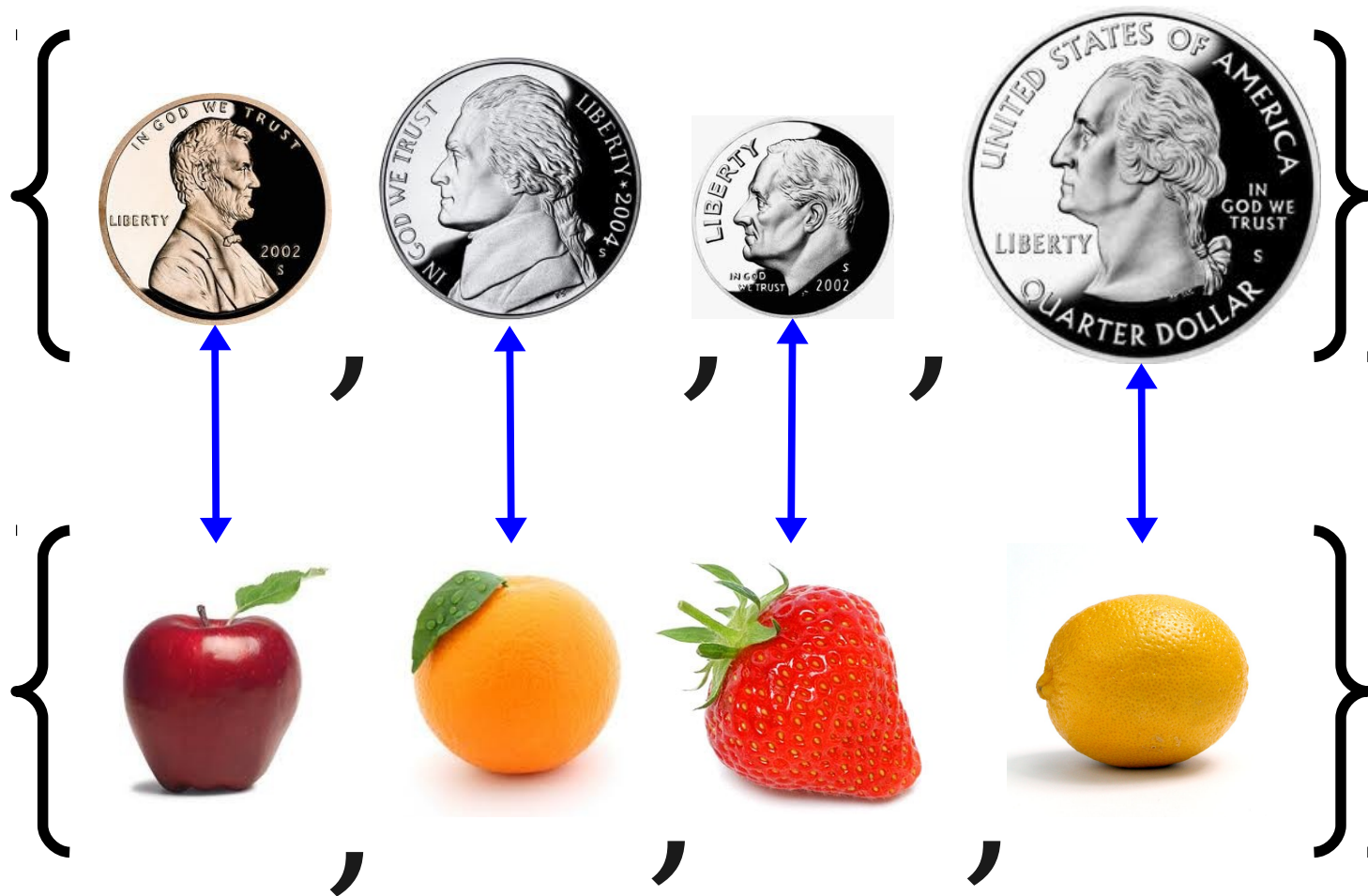
What is $|S|$?



How Big Are These Sets?

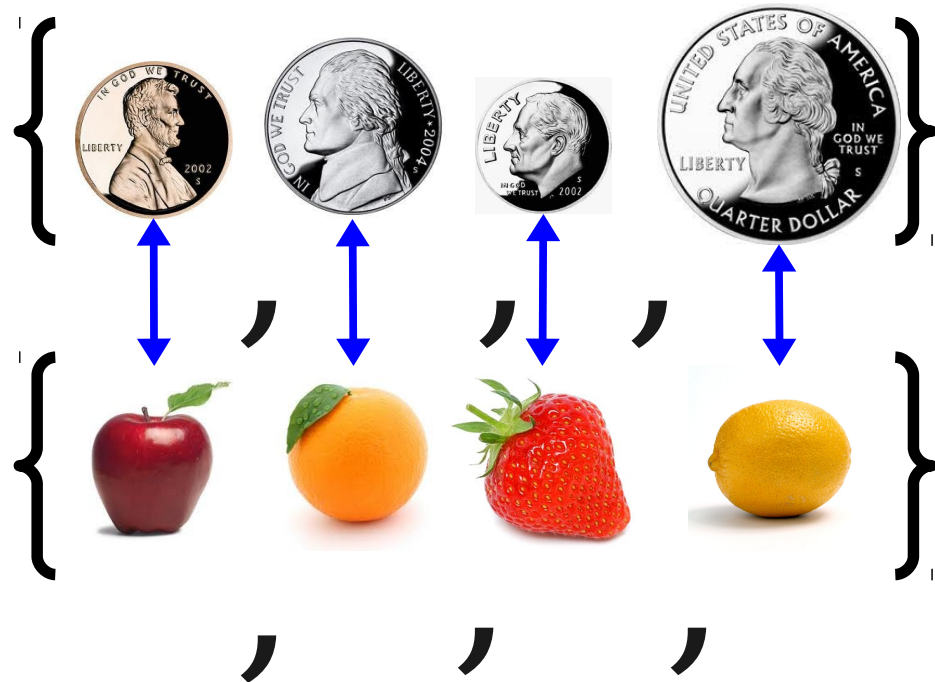


How Big Are These Sets?



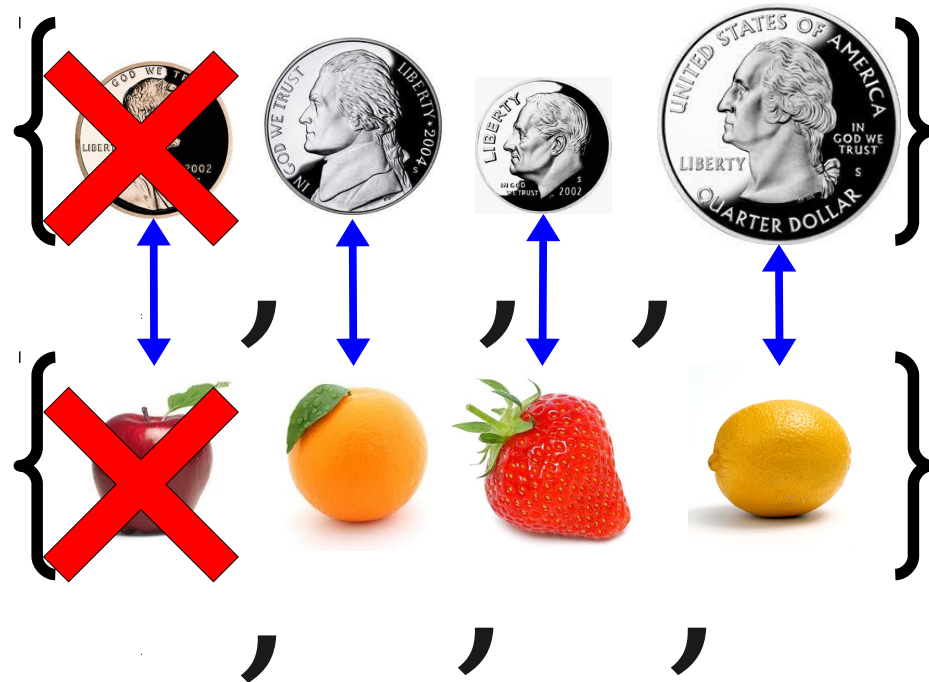
Comparing Cardinalities

- Two sets have the same cardinality if their elements can be put into a one-to-one correspondence with one another.
- The intuition:



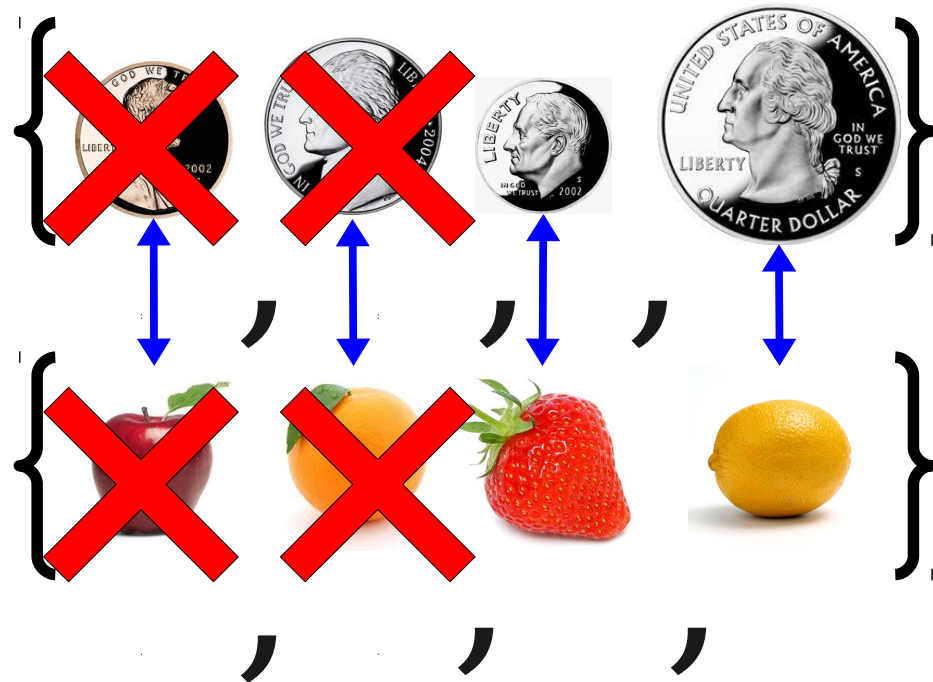
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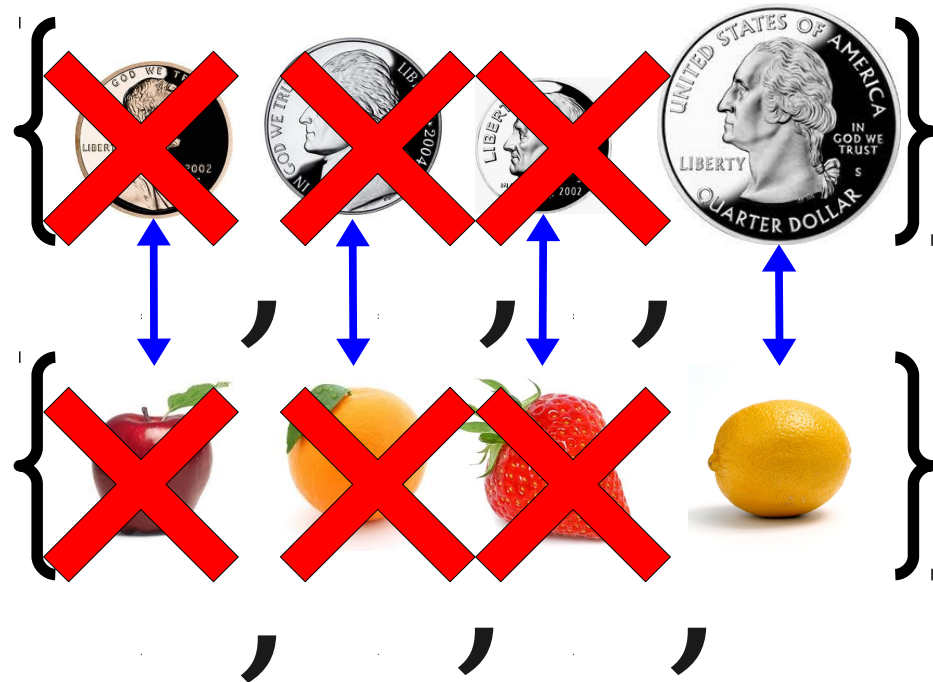
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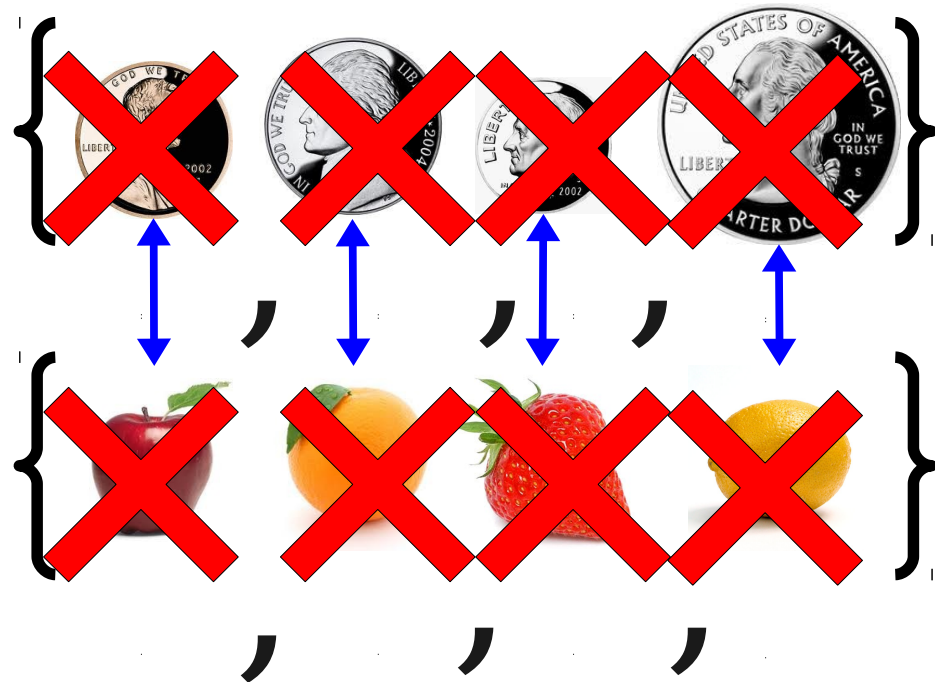
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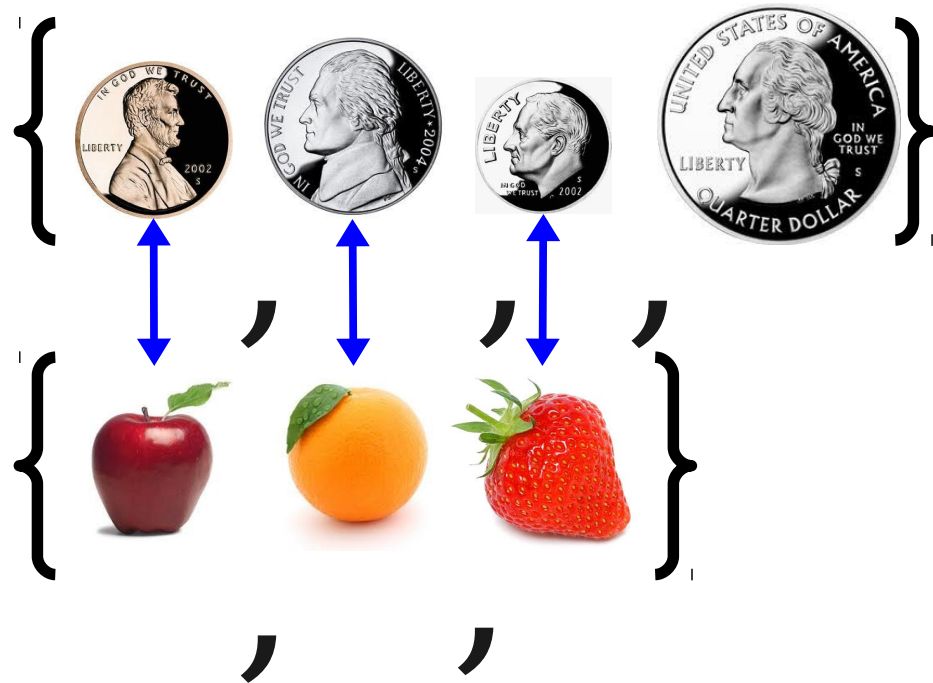
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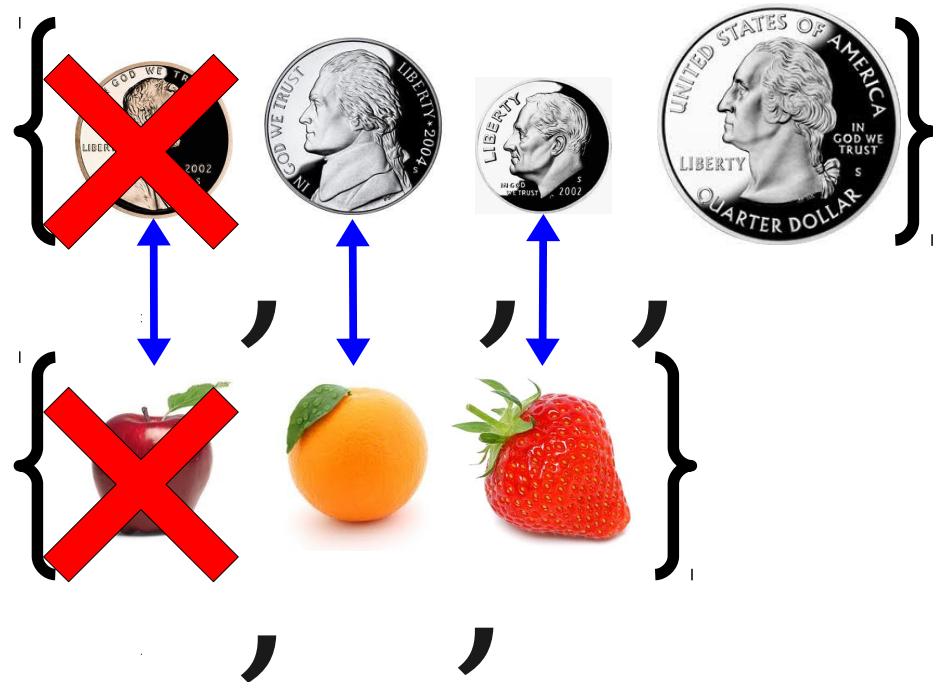
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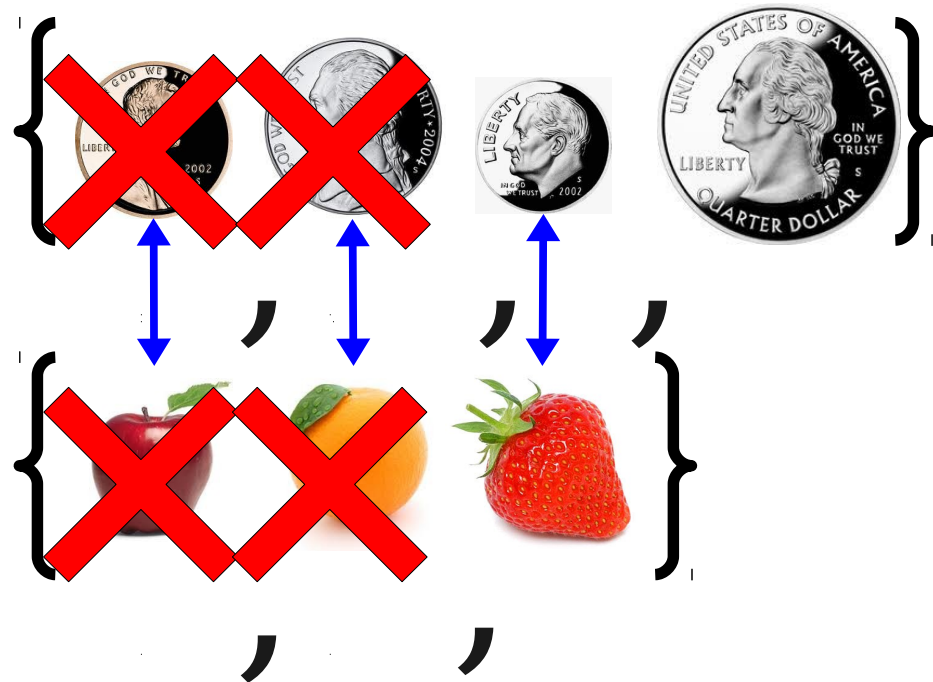
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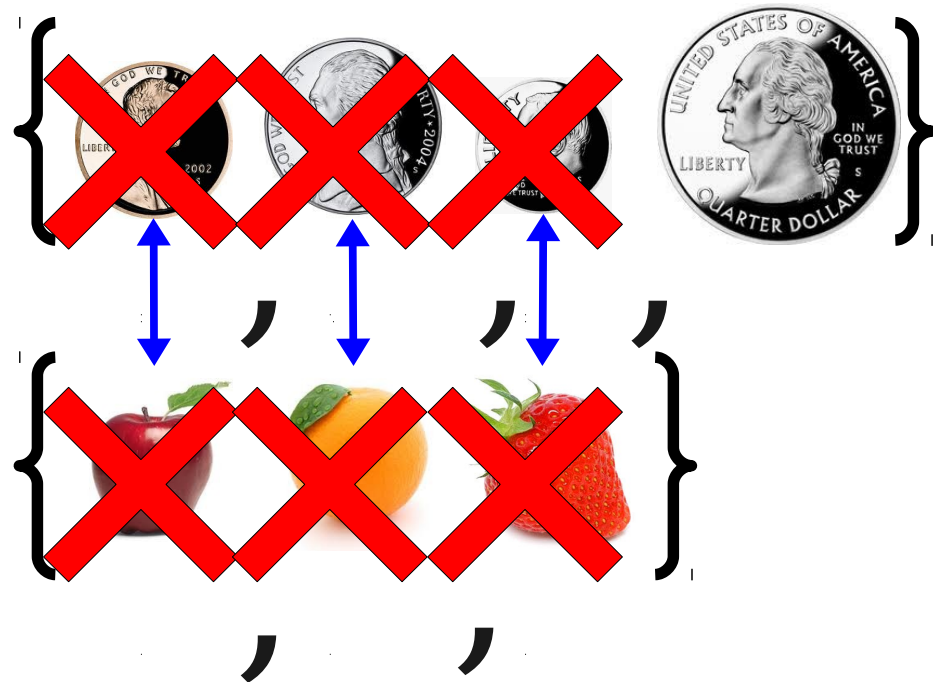
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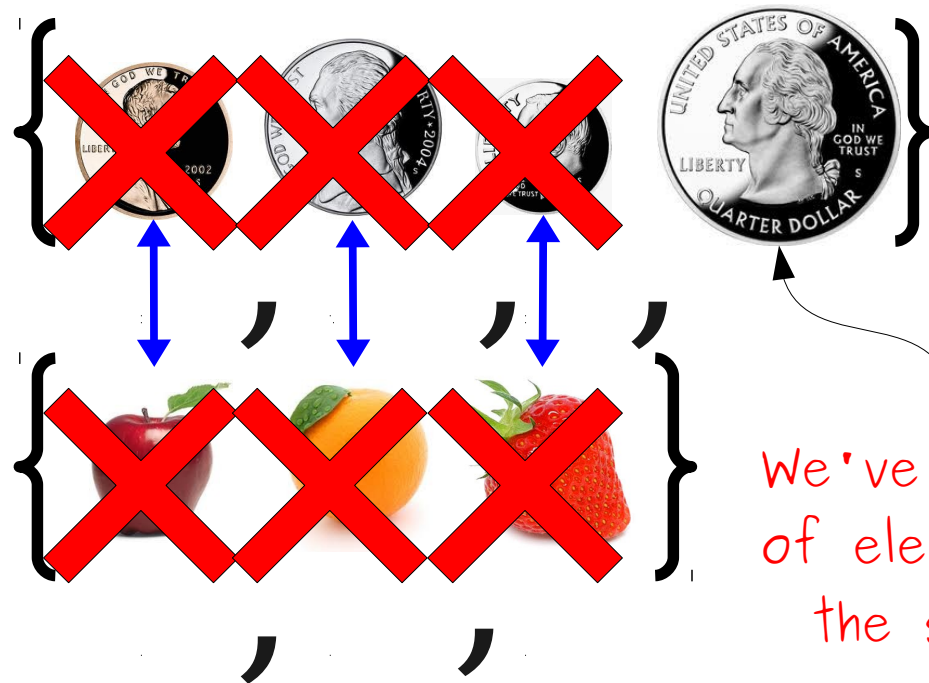
Comparing Cardinalities

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Comparing Cardinalities

- Two sets have the same cardinality if their elements can be put into a one-to-one correspondence with one another.
- The intuition:



We've run out
of elements in
the second
set!

Infinite Cardinalities

0 1 2 3 4 5 6 7 8 ...

0 2 4 6 8 10 12 14 16 ...

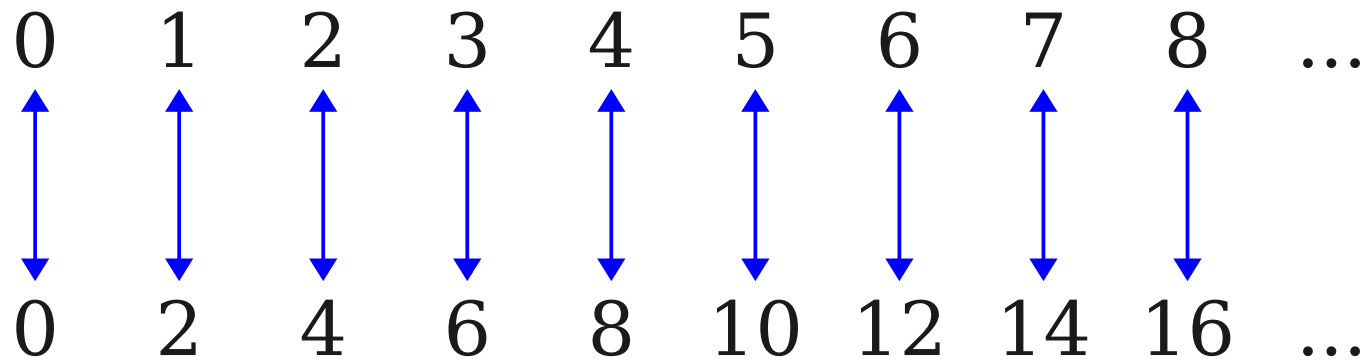
Infinite Cardinalities

0 1 2 3 4 5 6 7 8 ...

0 2 4 6 8 10 12 14 16 ...

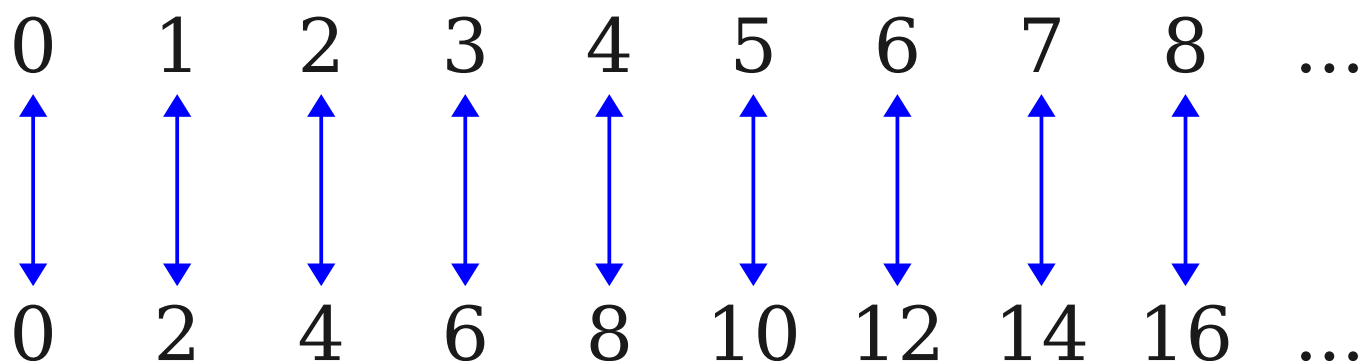
$$n \leftrightarrow 2n$$

Infinite Cardinalities



$$n \leftrightarrow 2n$$

Infinite Cardinalities



$$n \leftrightarrow 2n$$

$$S = \{ x \mid x \in \mathbb{N} \text{ and } x \text{ is even} \}$$

$$|S| = |\mathbb{N}| = \aleph_0$$

Infinite Cardinalities

\mathbb{N} 0 1 2 3 4 5 6 7 8 ...

\mathbb{Z} ... -3 -2 -1 0 1 2 3 4 ...

Infinite Cardinalities

\mathbb{N} 0 1 2 3 4 5 6 7 8 ...

\mathbb{Z} 0 1 -1 2 -2 3 -3 4 -4 ...

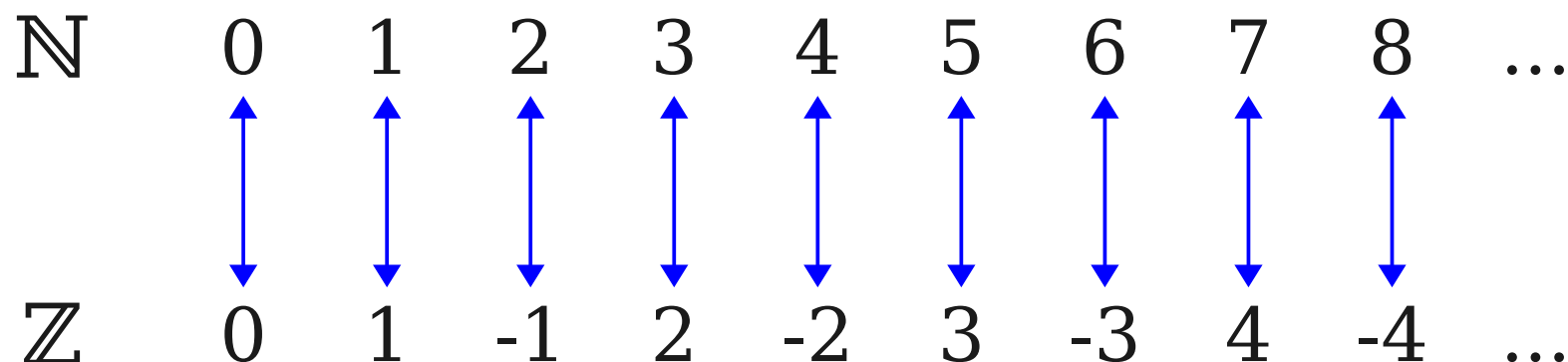
Infinite Cardinalities

\mathbb{N} 0 1 2 3 4 5 6 7 8 ...

\mathbb{Z} 0 1 -1 2 -2 3 -3 4 -4 ...

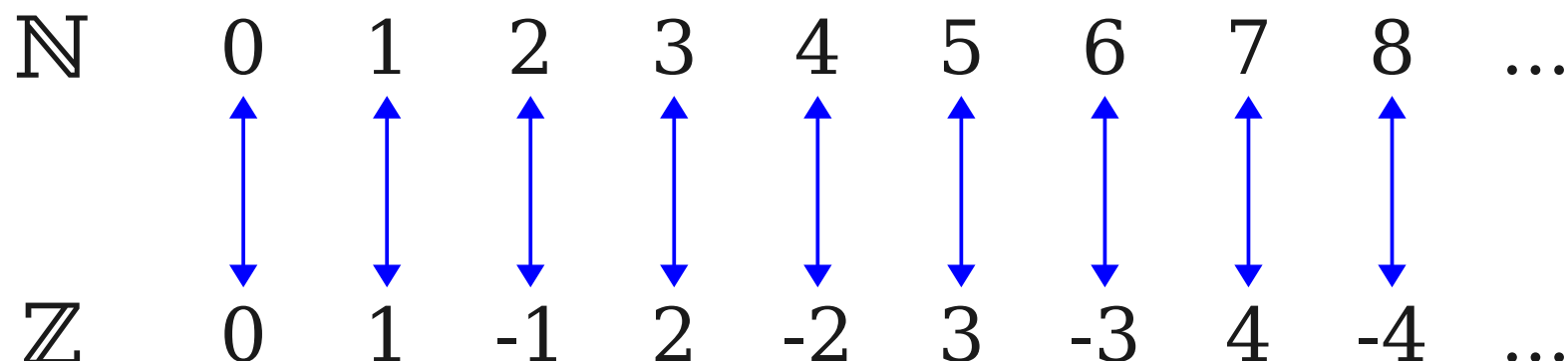
$n \leftrightarrow$ if n is even, then $-n / 2$
if n is odd, then $(n + 1) / 2$

Infinite Cardinalities



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Infinite Cardinalities



$n \leftrightarrow$ if n is even, then $-n / 2$
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$$|\mathbb{Z}| = |\mathbb{N}| = \aleph_0$$

Important Question

Do all infinite sets have
the same cardinality?

Prepare for one of the most beautiful (and surprising!) proofs in mathematics...

$$S = \left\{ \text{Lincoln Penny}, \text{Lincoln Cent} \right\}$$

$$\wp(S) = \left\{ \emptyset, \left\{ \text{Lincoln Cent} \right\}, \left\{ \text{Lincoln Penny} \right\}, \left\{ \text{Lincoln Penny}, \text{Lincoln Cent} \right\} \right\}$$

$$|S| < |\wp(S)|$$

$$S = \left\{ \text{Lincoln Penny}, \text{Kennedy Half Dollar}, \text{Button} \right\}$$

$$\wp(S) = \left\{ \begin{array}{l} \emptyset, \{ \text{Lincoln Penny} \}, \{ \text{Kennedy Half Dollar} \}, \{ \text{Button} \}, \\ \{ \text{Lincoln Penny}, \text{Kennedy Half Dollar} \}, \{ \text{Lincoln Penny}, \text{Button} \}, \{ \text{Kennedy Half Dollar}, \text{Button} \}, \\ \{ \text{Lincoln Penny}, \text{Kennedy Half Dollar}, \text{Button} \} \end{array} \right\}$$

$$|S| < |\wp(S)|$$

$$S = \{a, b, c, d\}$$

$$\wp(S) = \{$$

$$\emptyset,$$

$$\{a\}, \{b\}, \{c\}, \{d\},$$

$$\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{b, e\}$$

$$\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\},$$

$$\{a, b, c, d\}$$

$$\}$$

$$|S| < |\wp(S)|$$

If S is infinite, what is
the relation between $|S|$ and $|\wp(S)|$?

Does $|S| = |\wp(S)|$?

If $|S| = |\wp(S)|$, there has to be a one-to-one correspondence between elements of S and subsets of S .

What might this correspondence look like?

Infinite Cardinalities

- Recall: $|\mathbb{N}| = \aleph_0$.
- By Cantor's Theorem:

$$|\mathbb{N}| < |\wp(\mathbb{N})|$$

$$|\wp(\mathbb{N})| < |\wp(\wp(\mathbb{N}))|$$

$$|\wp(\wp(\mathbb{N}))| < |\wp(\wp(\wp(\mathbb{N})))|$$

$$|\wp(\wp(\wp(\mathbb{N})))| < |\wp(\wp(\wp(\wp(\mathbb{N}))))|$$

...

- **Not all infinite sets have the same size.**
- **There are multiple different infinities.**

What does this have to do
with computation?

“The set of all computer programs”

“The set of all problems to solve”

Strings and Problems

- Consider the set of all strings:
 $\{ "", "a", "b", "c", \dots, "aa", "ab", "ac," \dots \}$
- For any set of strings S , we can solve the following problem about S :
Write a program that accepts as input a string, then prints out whether or not that string belongs to set S .
- Therefore, there are at least as many problems to solve as there are sets of strings.

Every computer program is a string.

So, there can't be any more
programs than there are strings.

From Cantor's Theorem, we know that there are
more sets of strings than strings.

There are at least as many problems
as there are sets of strings.

$$|\mathbf{Programs}| \leq |\mathbf{Strings}| < |\mathbf{Sets\ of\ Strings}| \leq |\mathbf{Problems}|$$

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From Cantor's Theorem, we know that there are
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There are at least as many problems
as there are sets of strings.

|Programs| < |Problems|

**There are more
problems to solve than
there are programs to
solve them.**

It Gets Worse

- Because there are more problems than strings, we can't even *describe* some of the problems that we can't solve.
- Using more advanced set theory, we can show that there are *infinitely more* problems than solutions.
- In fact, if you pick a totally random problem, the probability that you can solve it is *zero*.

But then it gets better...

Where We're Going

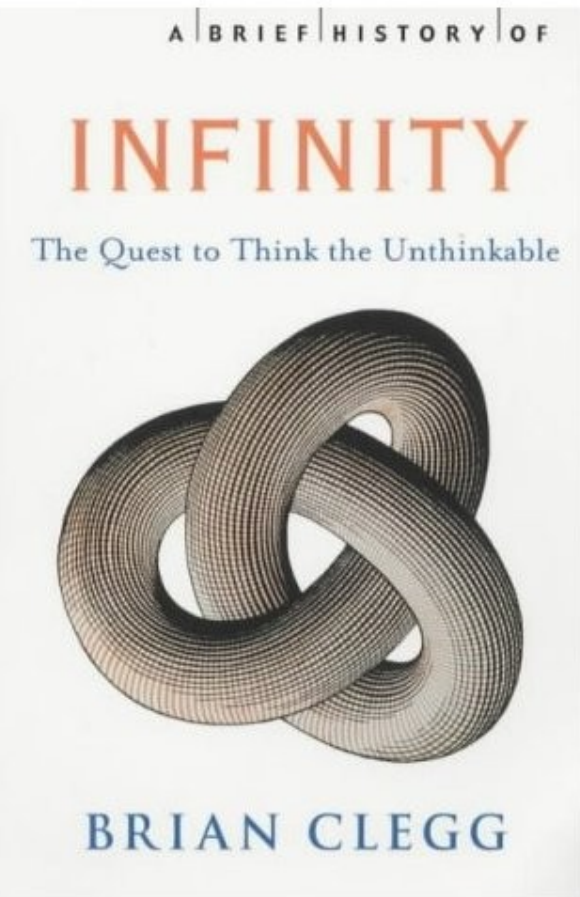
- **Given this hard theoretical limit, what *can* we compute?**
 - What are the hardest problems we *can* solve?
 - How powerful of a computer do we need to solve these problems?
 - Of what we can compute, what can we compute *efficiently*?
- **What tools do we need to reason about this?**
 - How do we build mathematical models of computation?
 - How can we reason about these models?

Next Time

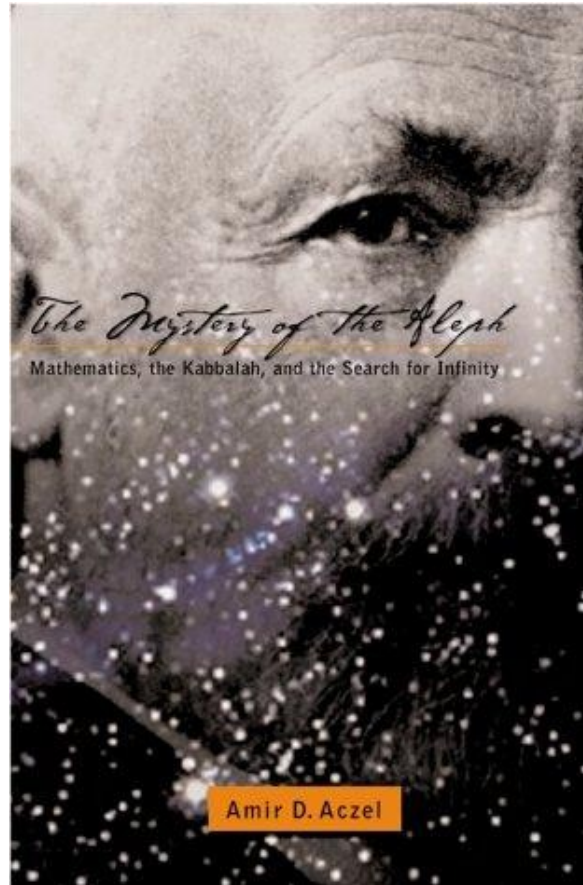
- **Mathematical Proof**
 - What is a mathematical proof?
 - How can we prove things with certainty?

Direct Proofs

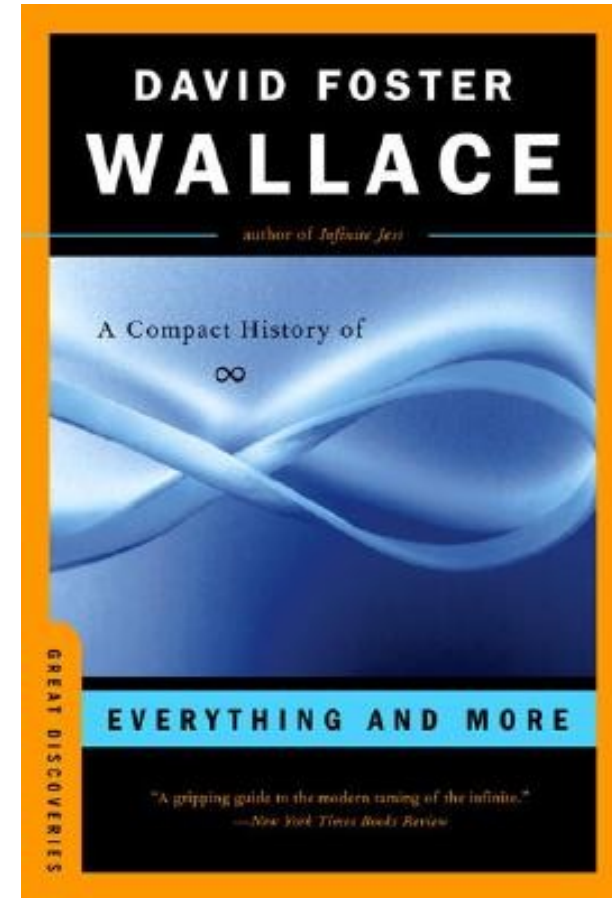
Recommended Reading



A Brief History of Infinity



The Mystery of the Aleph



Everything and More

What is a Proof?

Induction and Deduction

- In the sciences, much reasoning is done **inductively**.
 - Conduct a series of experiments and find a rule that explains all the results.
 - Conclude that there is a general principle explaining the results.
 - Even if all data are correct, the conclusion might be incorrect.
- In mathematics, reasoning is done **deductively**.
 - Begin with a series of statements assumed to be true.
 - Apply logical reasoning to show that some conclusion necessarily follows.
 - If all the starting assumptions are correct, the conclusion necessarily must be correct.

Structure of a Mathematical Proof

- Begin with a set of initial assumptions called **hypotheses**.
- Apply logical reasoning to derive the final result (the **conclusion**) from the hypotheses.
- Assuming that all intermediary steps are sound logical reasoning, the conclusion follows from the hypotheses.

Direct Proofs

Direct Proofs

- A **direct proof** is the simplest type of proof.
- Starting with an initial set of hypotheses, apply simple logical steps to prove the conclusion.
 - *Directly* proving that the result is true.
- Contrasts with **indirect proofs**, which we'll see on Friday.

Two Quick Definitions

- An integer n is **even** if there is some integer k such that $n = 2k$.
 - This means that 0 is even.
- An integer n is **odd** if there is some integer k such that $n = 2k + 1$.
- We'll assume the following for now:
 - Every integer is either even or odd.
 - No integer is both even and odd.

A Simple Direct Proof

Theorem: If n is even, then n^2 is even.

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This means that $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$.

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Since $2k^2$ is an integer, this means that there is some integer m (namely, $2k^2$) such that $n^2 = 2m$.

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Thus n^2 is even. ■

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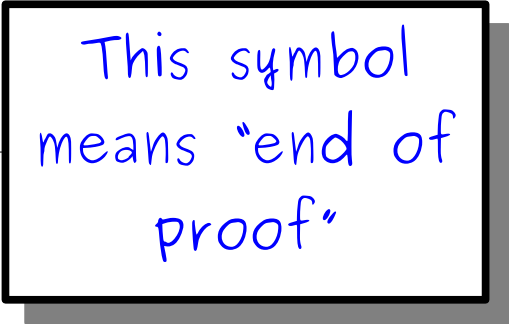
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This symbol
means "end of
proof"

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there is some integer m such that $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$.

Thus n^2 is even. ■

This is the definition of an even integer. When writing a mathematical proof, it's common to call back to the definitions.

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Since $2(2k^2)$ is even, there is some integer m such that $n^2 = 2m$.

Thus n^2 is even.

Notice how we use the value of k that we obtained above. Giving names to quantities, even if we aren't fully sure what they are, allows us to manipulate them. This is similar to variables in programs.

uch

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Our ultimate goal is to prove that n^2 is even. This means that we need to find some m such that $n^2 = 2m$. Here, we're explicitly showing how we can do that.

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Hey, that's what we were trying to show! We're done now.

uch

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Theorem: For any sets A , B , and C , if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

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How do we prove
that this is true for
any choice of sets?

Proving Something Always Holds

- Many statements have the form

For any X , $P(X)$ is true.

- Examples:

For all integers n , if n is even, n^2 is even.

For any sets A , B , and C , if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

For all sets S , $|S| < |\wp(S)|$.

- How do we prove these statements when there are infinitely many cases to check?

Arbitrary Choices

- To prove that $P(x)$ is true for all possible x , show that no matter what choice of x you make, $P(x)$ must be true.
- Start the proof by making an arbitrary choice of x :
 - “Let x be chosen arbitrarily.”
 - “Let x be an arbitrary even integer.”
 - “Let x be an arbitrary set containing 137.”
 - “Consider any x .”
- Demonstrate that $P(x)$ holds true for this choice of x .
- Conclude that since the choice of x was arbitrary, $P(x)$ must hold true for all choices of x .

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Proof: Let A , B , and C be arbitrary sets with $A \subseteq B$ and $B \subseteq C$.

We're showing here that regardless of what A , B , and C you pick, the result will still be true.

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To prove a statement of the form

“If P , then Q ”

Assume that **P** is true, then show that **Q** must be true as well.

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Proof: Consider an arbitrary even natural number, say, 16. 16 is even, and it has no odd divisors. Since our choice was arbitrary, for any arbitrary n , if n is even, n has no odd divisors. ■

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RPG RAPTOR SHARK

YOUR ARGUMENT IS INVALID

memegenerator.net

Proof: Consider $n = 16$. Since our n is even, if n is even

al number,
visors.
rbitrary

ar·bi·trar·y

adjective /'ärbi,trerē/

1. Based on random choice or personal whim, rather than any reason or system - *“his mealtimes were entirely arbitrary”*
2. (of power or a ruling body) Unrestrained and autocratic in the use of authority - *“arbitrary rule by King and bishops has been made impossible”*
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Not this
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Use this
definition

To prove something is true for all x , **do not** choose an x and base the proof off of your choice!

Instead, leave x unspecified and show that no matter what x is, the specified property must hold.

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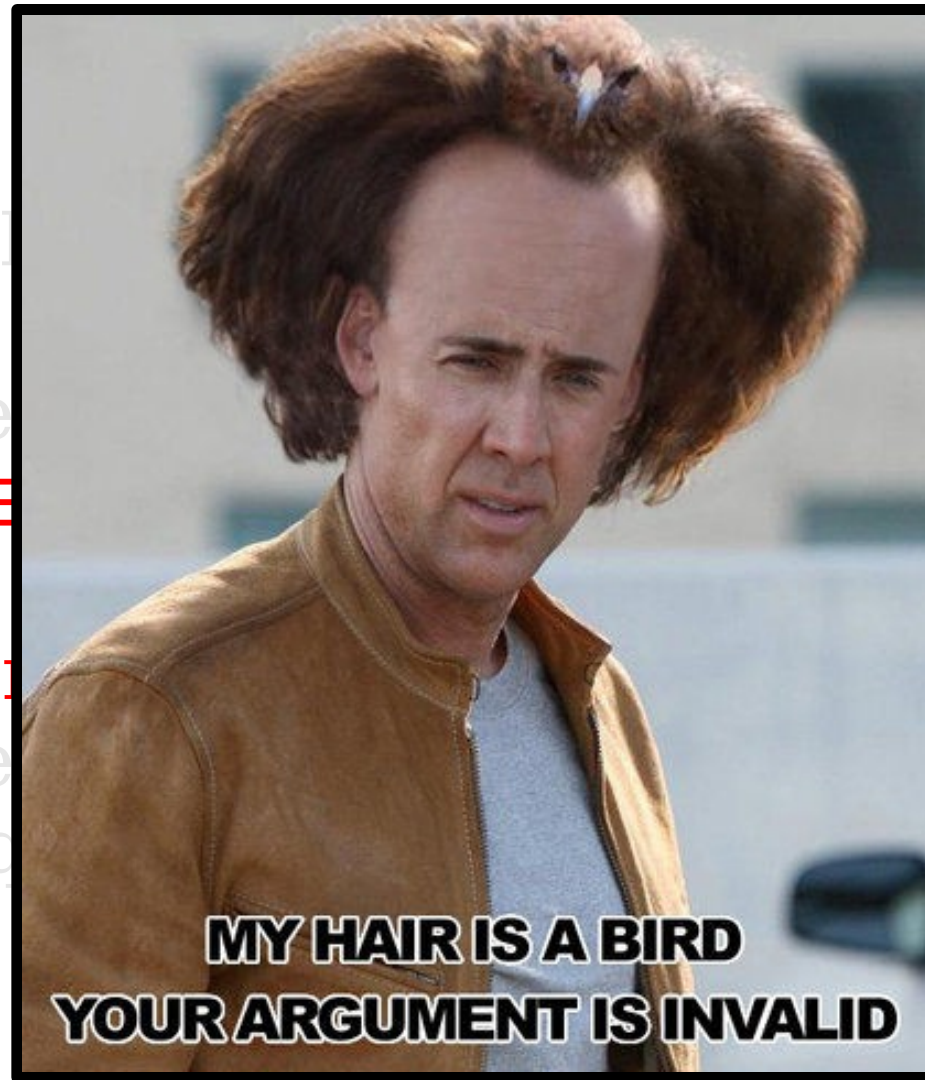
Consider any arbitrary $x \in A \cap B$. This means that $x \in A$ and $x \in B$, so $x \in A$ as required. ■

Another Incorrect Proof

Theorem: $A \cap B = A$.

Proof: We show $A \cap B \subseteq A$ and $A \subseteq A \cap B$.
Let $x \in A \cap B$. Then $x \in A$ and $x \in B$.
Conversely, let $x \in A$. Then $x \in A \cap B$. This shows $A \subseteq A \cap B$.
Therefore, $A \cap B = A$.

Conclusion: The proof is invalid because it assumes what it is trying to prove.



If you want to prove that P implies Q ,
assume P and prove Q .

Don't assume Q and then prove P !

An Entirely Different Proof

Theorem: There exists a natural number $n > 0$ such that the sum of all natural numbers less than n is equal to n .

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This is a fundamentally different type of proof that what we've done before. Instead of showing that every object has some property, we want to show that some object has a given property.

Universal vs. Existential Statements

- A **universal statement** is a statement of the form
For all x , $P(x)$ is true.
- We've seen how to prove these statements.

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- A **universal statement** is a statement of the form
For all x , $P(x)$ is true.
- We've seen how to prove these statements.
- An **existential statement** is a statement of the form
There exists an x for which $P(x)$ is true.
- How do you prove an existential statement?

Proving an Existential Statement

- We will see several different ways to prove “there is some x for which $P(x)$ is true.”
- Simple approach: Just go and find some x for which $P(x)$ is true!
 - In our case, we need to find a positive natural number n such that that sum of all smaller natural numbers is equal to n .
 - Can we find one?

An Entirely Different Proof

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An Entirely Different Proof

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Proof: Take $n = 3$.

There are three natural numbers smaller than 3: 0, 1, and 2.

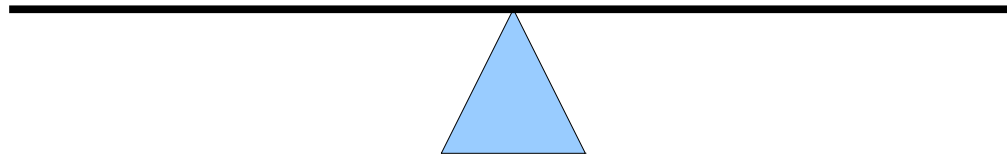
We have $0 + 1 + 2 = 3$.

Thus 3 is a natural number greater than zero equal to the sum of all smaller natural numbers. ■

The Counterfeit Coin Problem

Problem Statement

- You are given a set of three seemingly identical coins, two of which are real and one of which is counterfeit.
- The counterfeit coin weighs more than the rest of the coins.
- You are given a balance. Using only one weighing on the balance, find the counterfeit coin.



Theorem: Given three coins, one of which weighs more than the rest, and a balance, there is a way to find which coin is counterfeit in one weighing.

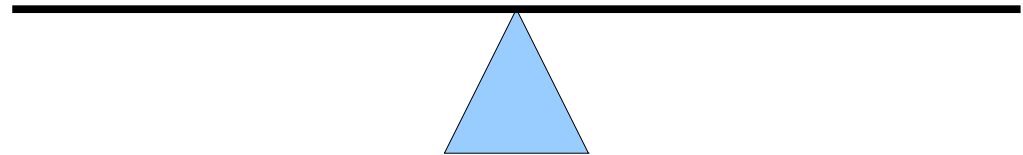
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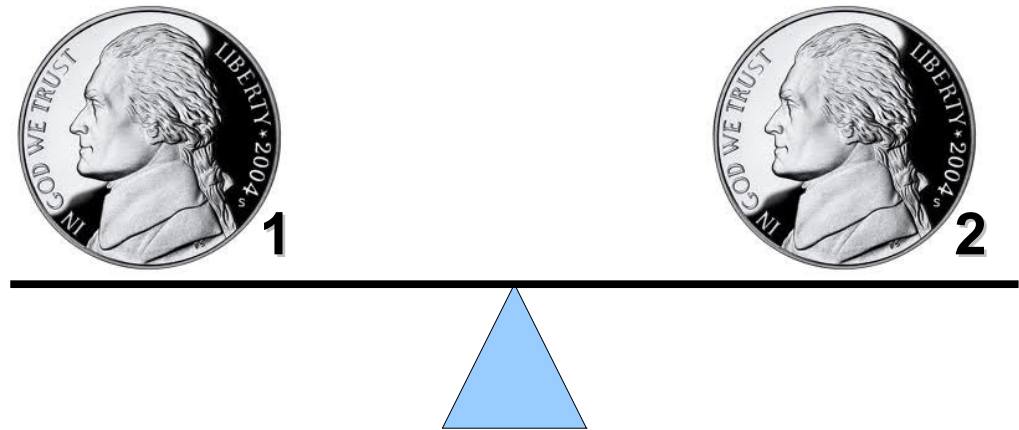
This is an existential statement.

We should try to look for an actual way to do this.

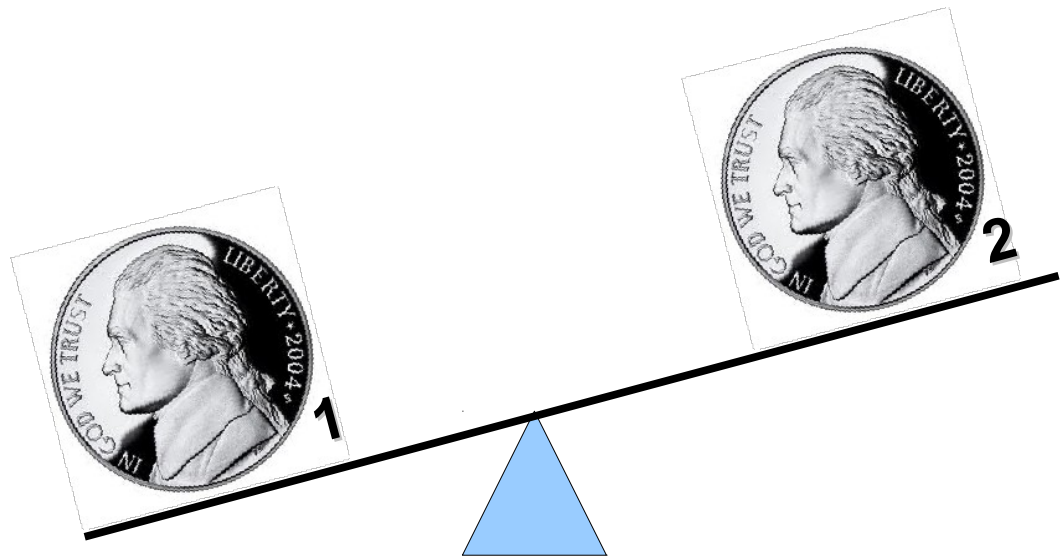
Finding the Counterfeit Coin



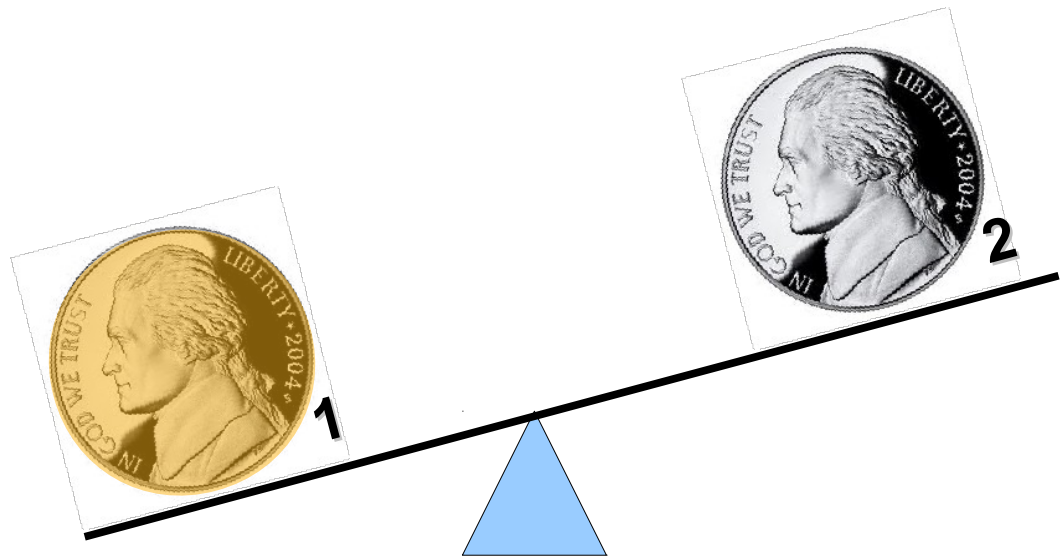
Finding the Counterfeit Coin



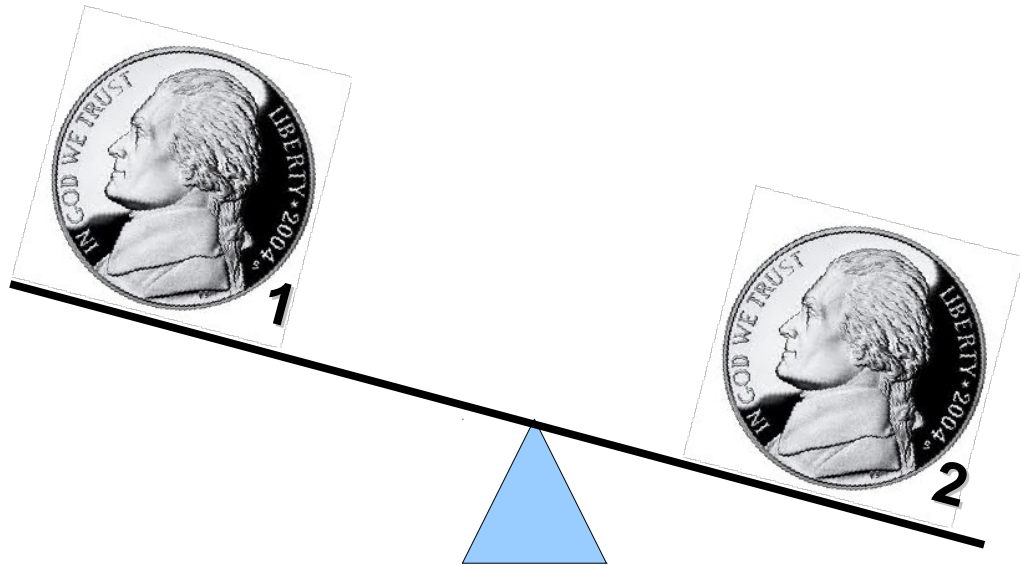
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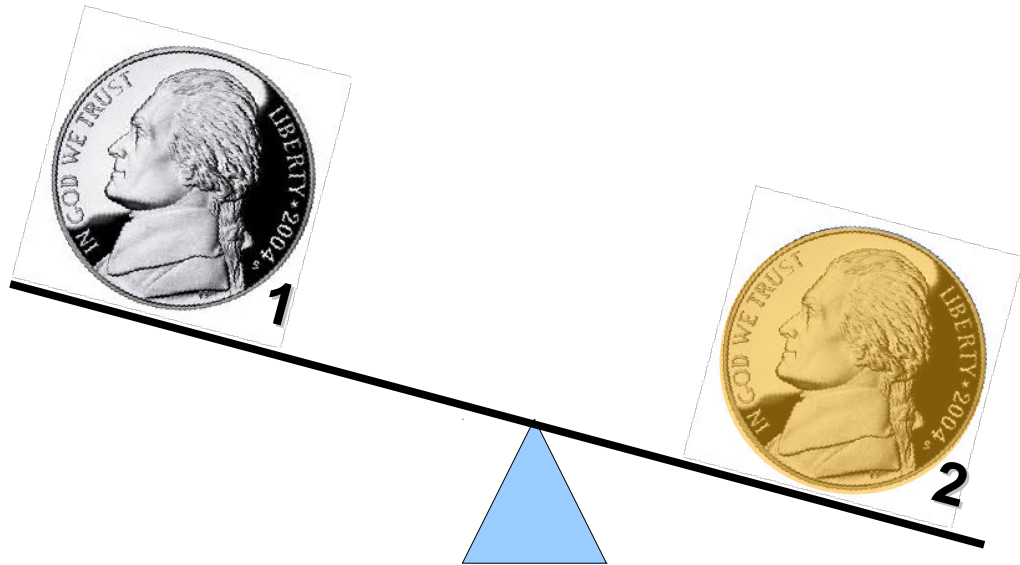
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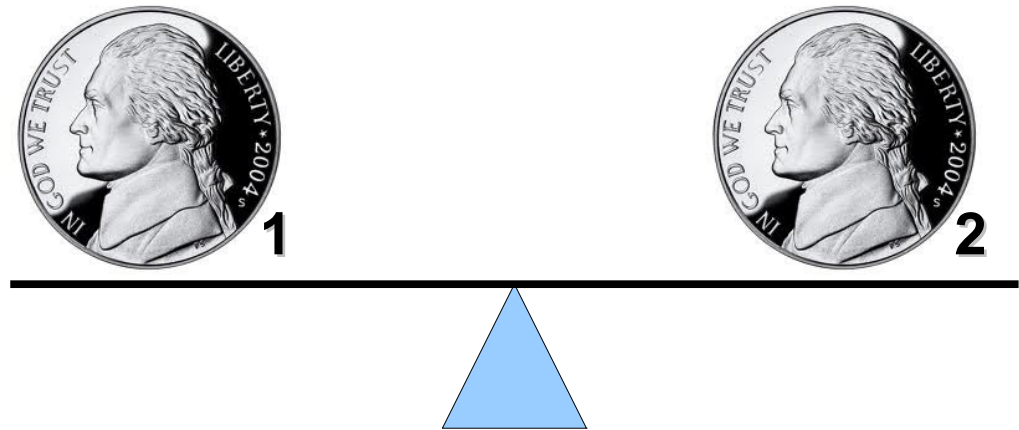
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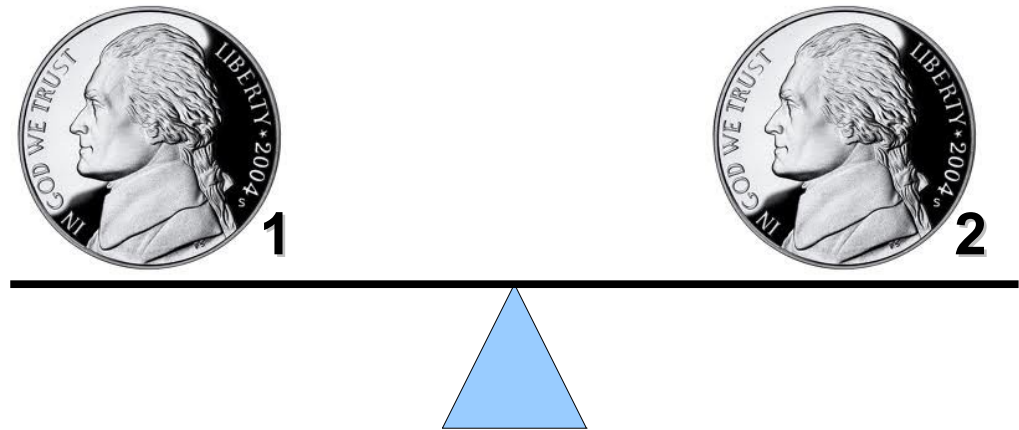
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Finding the Counterfeit Coin



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Case 3: C This is called a *proof by cases* (alternatively, a *proof by exhaustion*) and works by showing that the theorem is true regardless of what specific outcome arises.

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In a proof by cases, after demonstrating each case, you should summarize the cases afterwards to make your point clearer.

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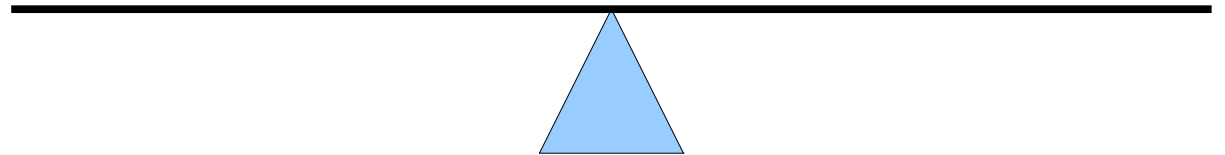
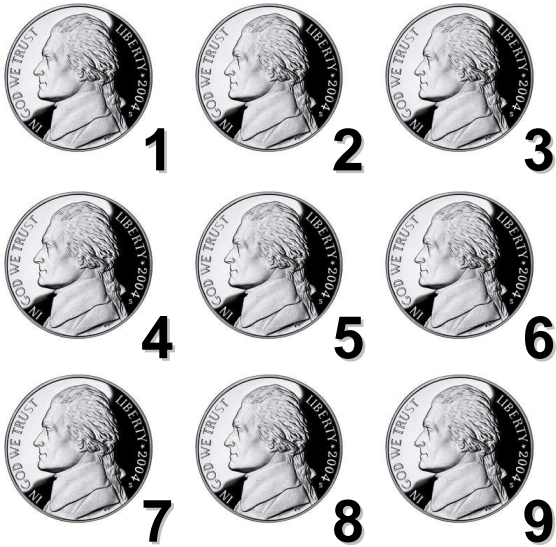
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A Harder Problem

- You are given a set of **nine** seemingly identical coins, eight of which are real and one of which is counterfeit.
- The counterfeit coin weighs more than the rest of the coins.
- You are given a balance. Using only **two** weighings on the balance, find the counterfeit coin.

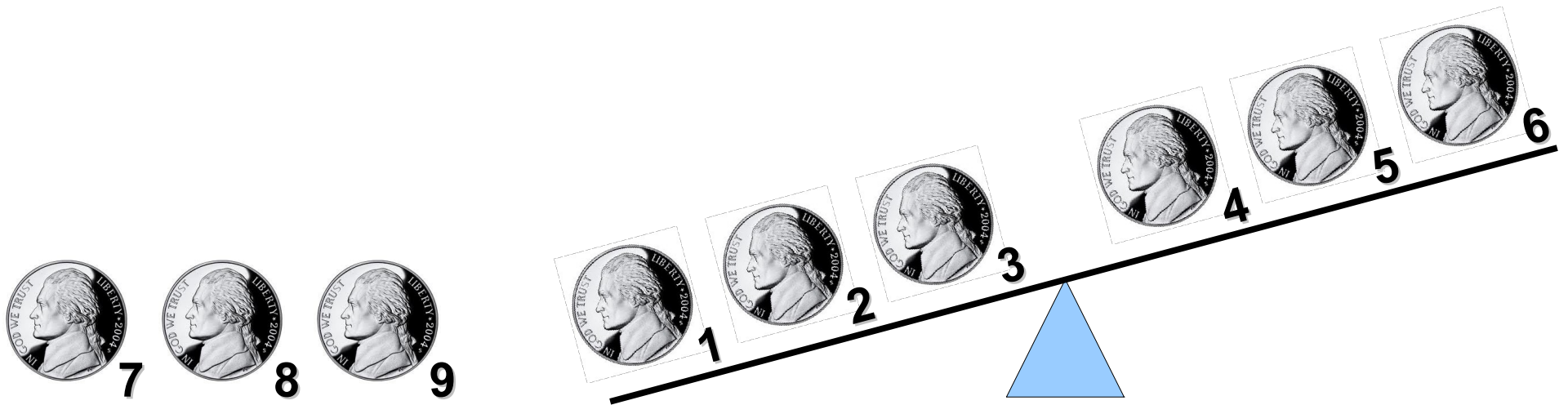
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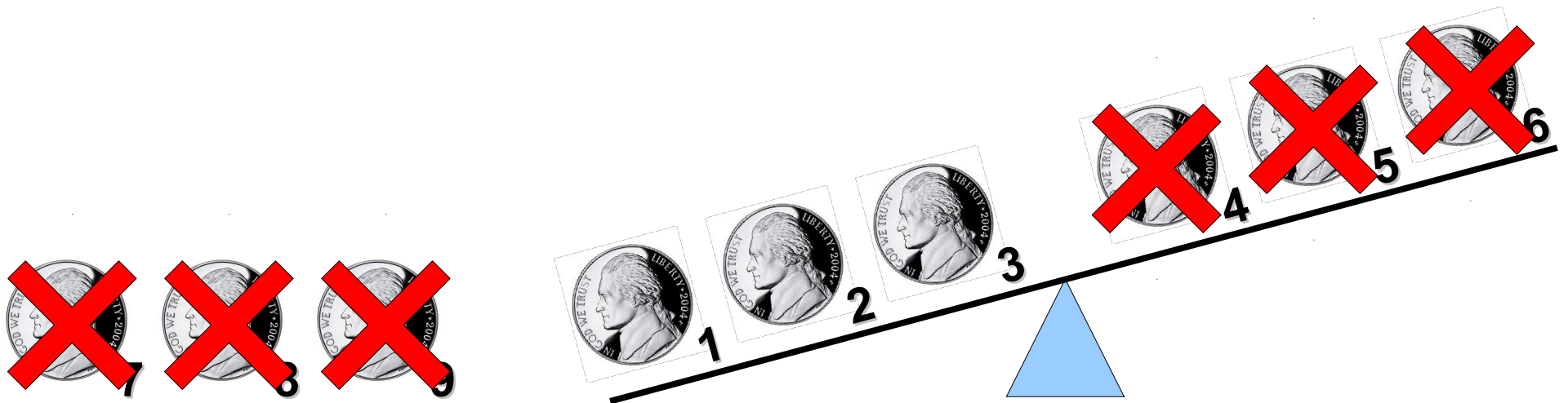
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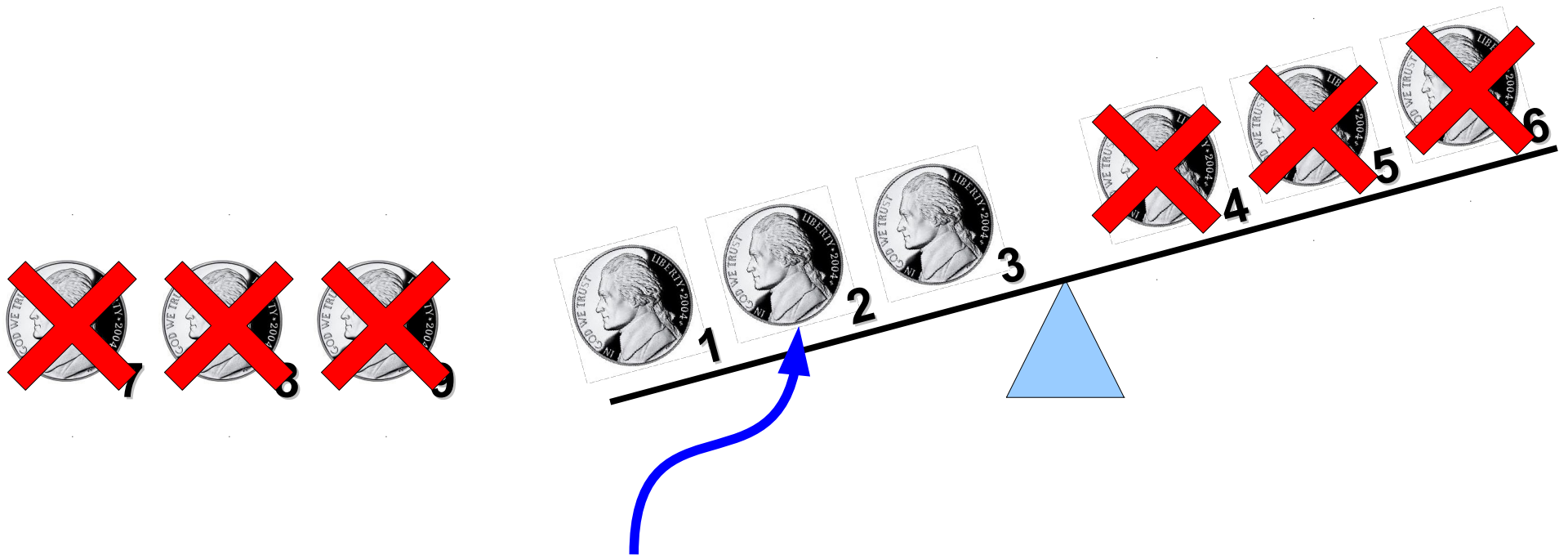
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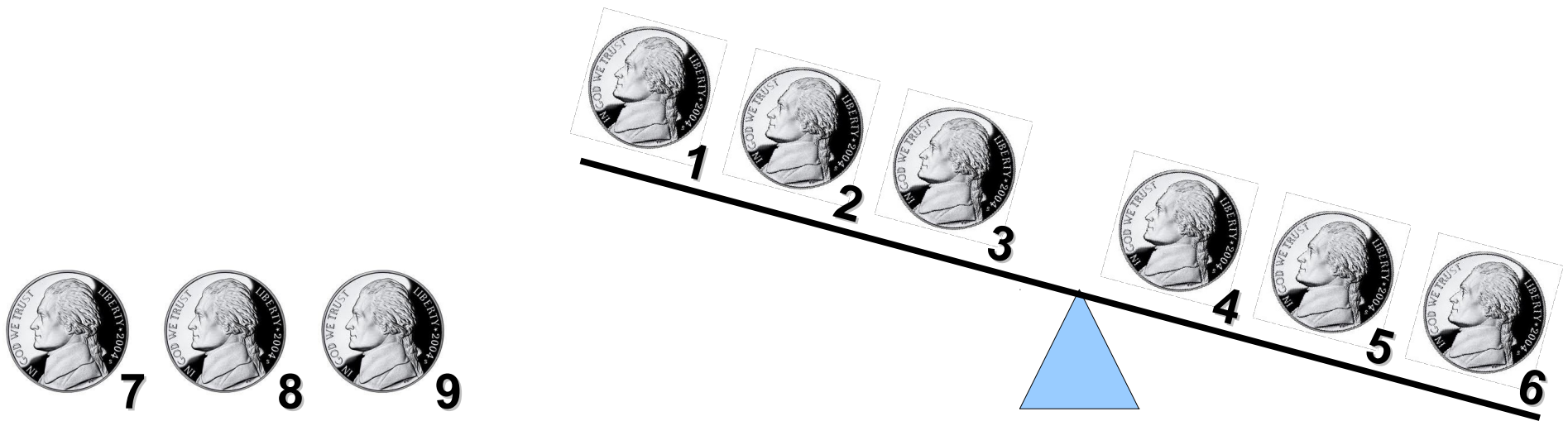


Finding the Counterfeit Coin

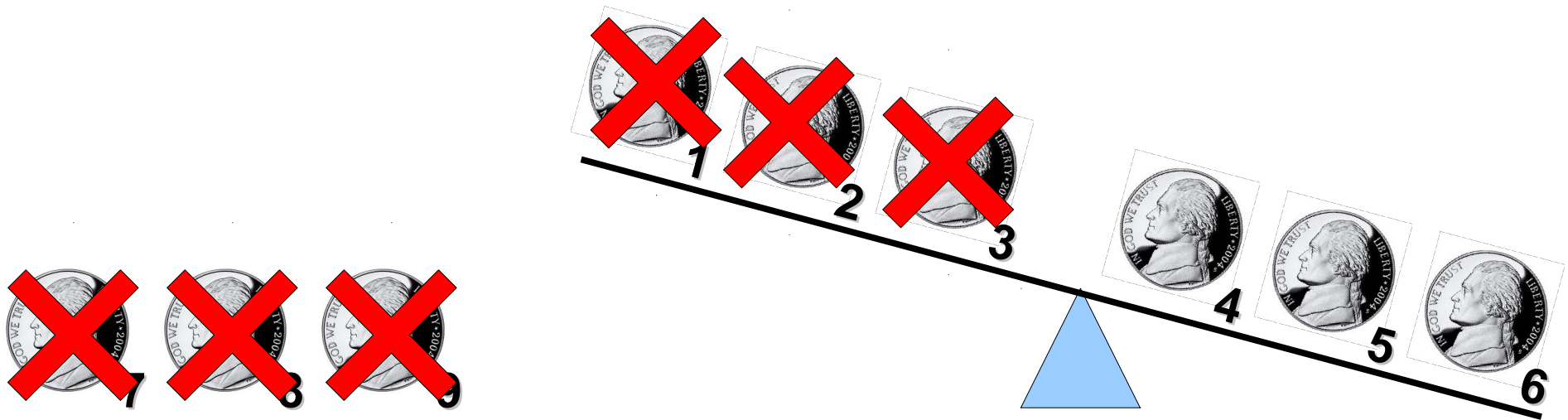


Now we have one weighing to find
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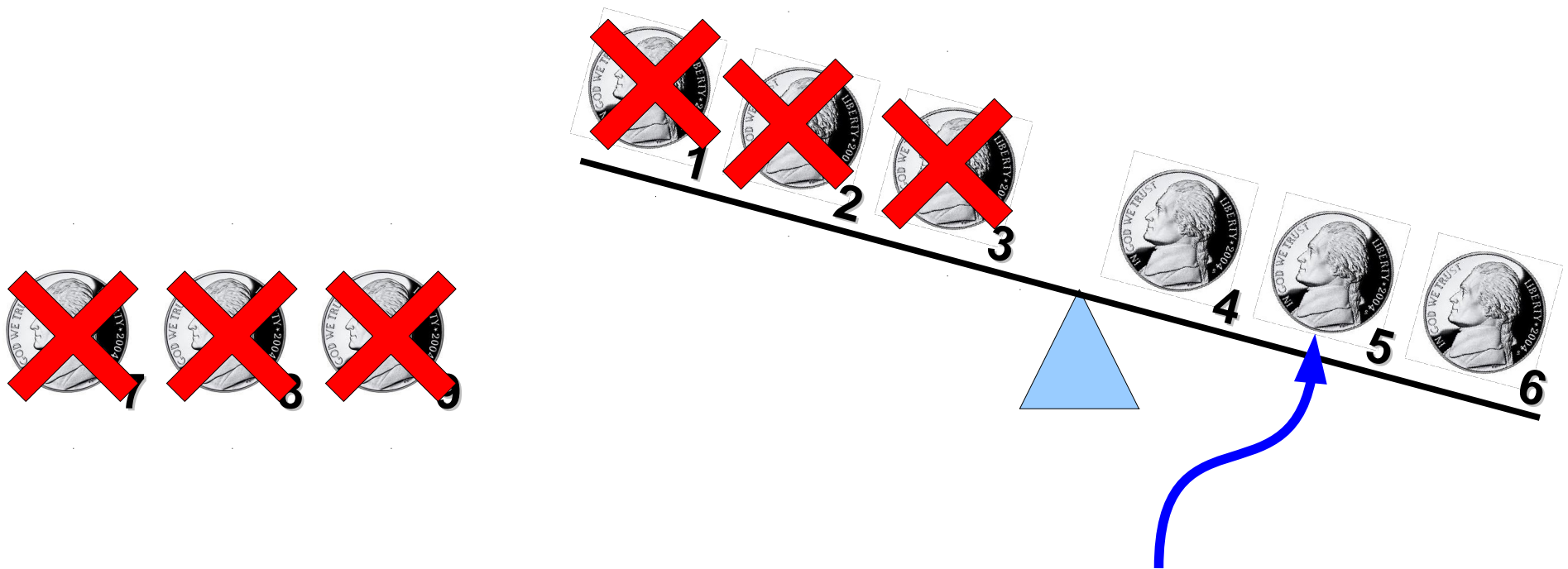
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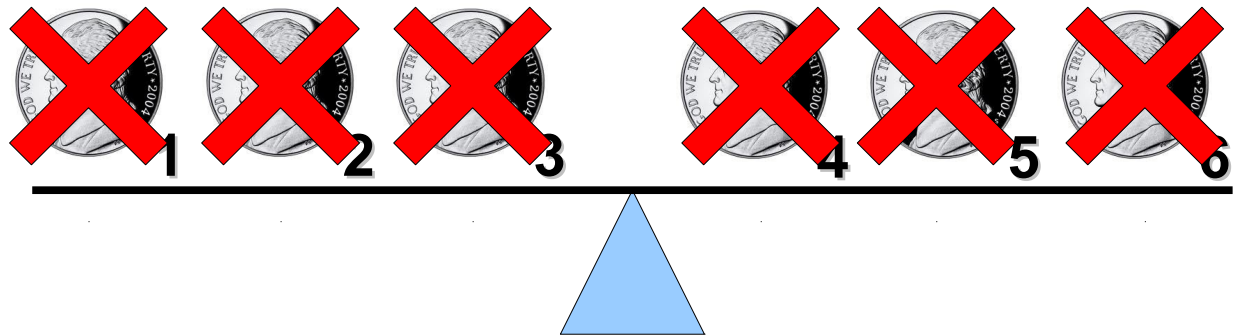


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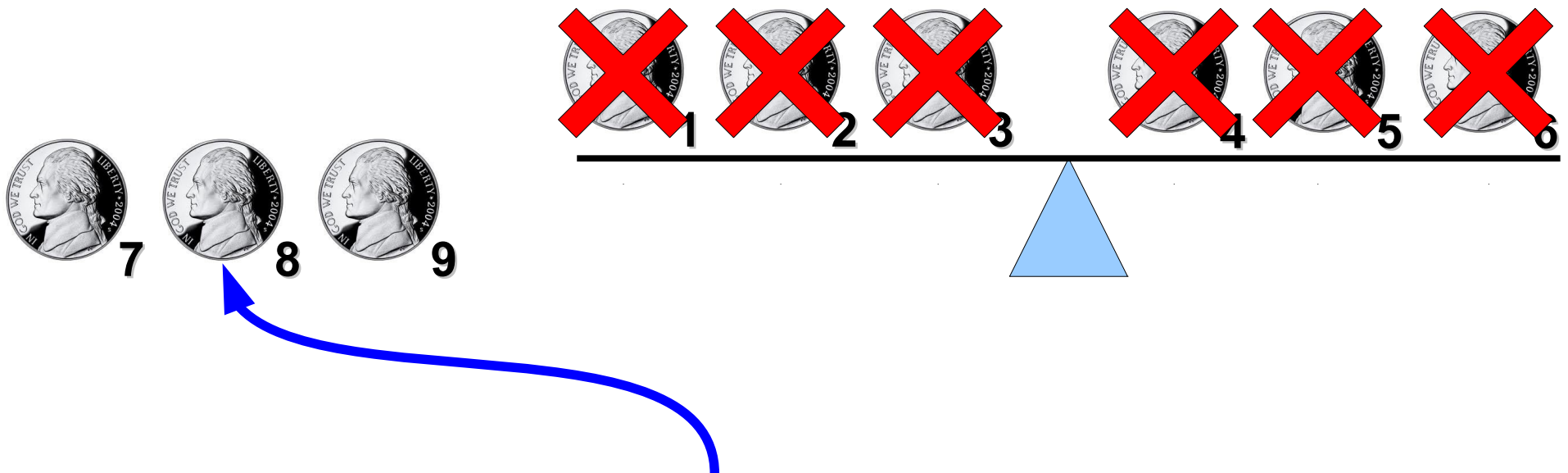
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When proving a result, it's perfectly fine to refer to theorems you've proven earlier! Here, we cite our theorem from before and say it's possible to find which of three coins is the counterfeit.

In this course, feel free to refer to any theorem that we've proven in lecture, in the course notes, in the book, in section, or in previous problem sets when writing your proofs.

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Relations Between Proofs

- Proofs often build off of one another: large results are almost often accomplished by building off of previous work.
 - Like writing a large program – split the work into smaller methods, across different classes, etc. instead of putting the whole thing into **main**.
- A result that is proven specifically as a stepping stone toward a larger result is called a **lemma**.
- We can treat the proof of the three-coin case as a lemma in the larger proof about nine coins.
 - The result in itself isn't particularly impressive, but it helps us prove a more advanced result.

Our Very Second Lemma

- Set equality is defined as follows

**$A = B$ precisely when
for every $x \in A$, $x \in B$ and vice-versa.**

- This definition makes it a bit tricky to prove that two sets are equal.
- Instead, we will prove the following result:

**For any sets A and B ,
if $A \subseteq B$ and $B \subseteq A$, then $A = B$.**

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Thus whenever $x \in A$, $x \in B$ and whenever $x \in B$, $x \in A$ as well.

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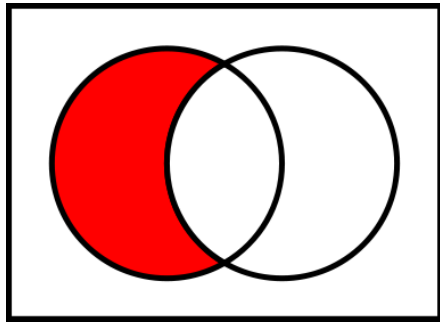
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Using Our Lemma

- We can use this lemma to prove properties of how sets relate to one another.
- For example, let's prove that $(A - B) \cup B = A \cup B$.

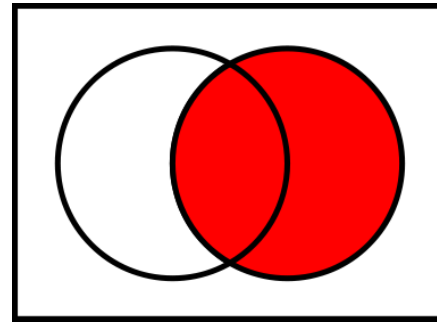
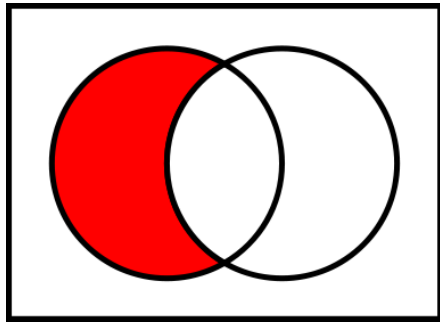
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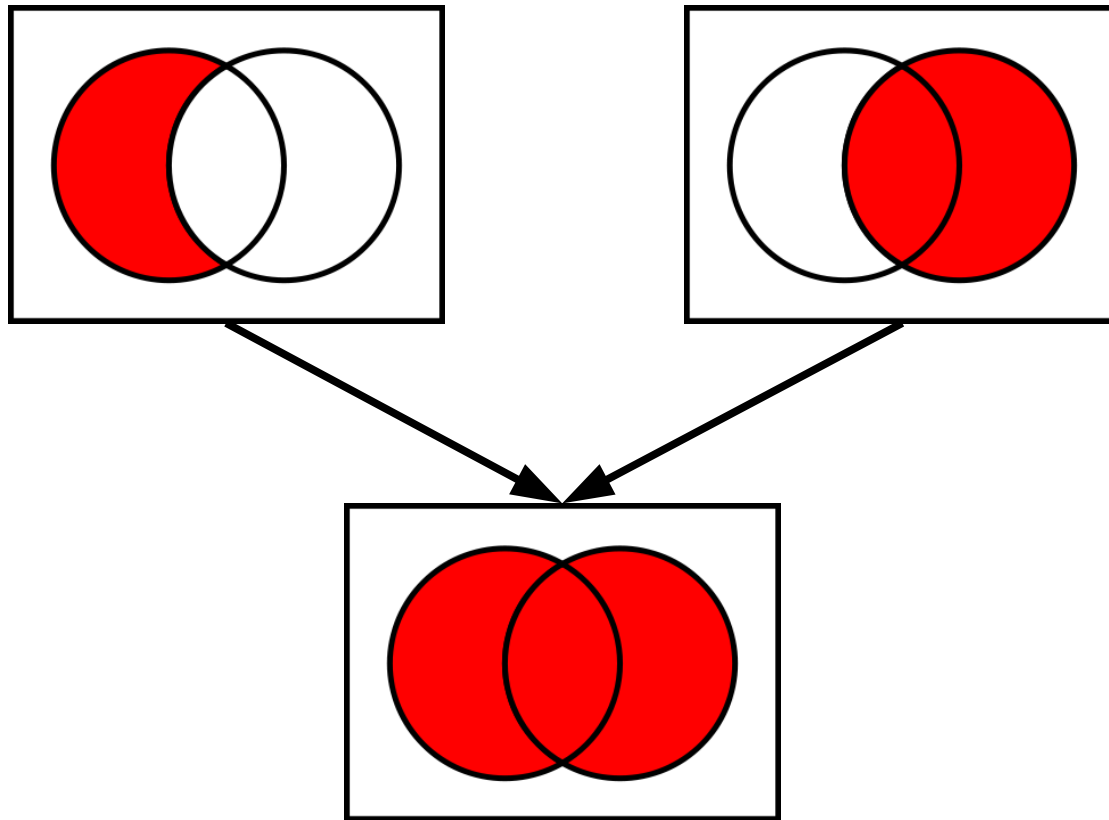
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- For example, let's prove that $(A - B) \cup B = A \cup B$.
- Proof idea: Show that each set is a subset of the other.

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Proof: Let A and B be arbitrary sets. Consider any $x \in (A - B) \cup B$.

By definition, $(A - B) \cup B$ is the set of all x where $x \in A - B$ or $x \in B$, so we have that $x \in A - B$ or $x \in B$. We consider these two cases separately:

Case 1: $x \in A - B$. By definition, $A - B$ is the set of all x where $x \in A$ and $x \notin B$. This means that $x \in A$, and so $x \in A \cup B$ as well.

Case 2: $x \in B$. Then $x \in A \cup B$ as well.

In either case, any $x \in (A - B) \cup B$ also satisfies $x \in A \cup B$, so $(A - B) \cup B \subseteq A \cup B$ as required. ■

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We're already doing a proof by cases, but inside this particular case there's two more cases to consider. There's nothing wrong with that; just like nested loops in a program, we can have nested cases.

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In either case, any $x \in (A - B) \cup B$ also satisfies $x \in A \cup B$, so $(A - B) \cup B \subseteq A \cup B$ as required.

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In either case, any $x \in (A - B) \cup B$ also satisfies $x \in A \cup B$, so $(A - B) \cup B \subseteq A \cup B$ as required. ■

Theorem: For any sets A and B , $(A - B) \cup B = A \cup B$.

Proof: Let A and B be arbitrary sets.

By Lemma 1, $(A - B) \cup B \subseteq A \cup B$.

By Lemma 2, $A \cup B \subseteq (A - B) \cup B$.

Consequently, by our earlier lemma,
 $(A - B) \cup B = A \cup B$. ■

Next Time

- Indirect Proofs
 - Proof by contradiction.
 - Proof by contrapositive.